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RESEARCH PAPER

The Superiority of the EGARCH-Odd Exponentiated Skew-t Model in Predicting Financial Returns Volatility

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Abstract
The key point for volatility forecast is to utilize appropriate innovation conditional density. The specification of flexible innovation density is very essential given that it directly affects the accuracy of volatility prediction. In this work, a new odd exponentiated skew-t (OE_{ST}) innovation density is introduced for exponentiated generalized autoregressive conditional heteroscedasticity (EGARCH) models for modeling the daily volatility of financial return series. The simulation via Monte Carlo experiment indicates that the estimators compared are asymptotically unbiased and consistent given that their biases converge to zero as the sample size increases. The maximum likelihood and maximum product of spacing procedures dominate the other procedures. The real dataset application based on the First Bank Nigeria shock price index is given to show the performance of the EGARCH model specified under OE_{ST} innovation density relative to normal, student-t, generalized error, skew-normal, skew student-t, skew generalized error, generalized hyperbolic and Johnson reparametrized densities in terms of volatility accuracy. Overall, the empirical results show that the EGARCH model with OE_{ST} innovation density generates better in-and out-of-sample performance than all the other models.

Keywords: EGARCH Model, Estimation Methods, Financial Returns, Innovations Density, Monte-Carlo Simulation.
JEL Classification: C13, C15, C40, C46, G17.

1. Introduction
Asset-holding financial institutions, stock firms and investment portfolio managers' significant wellsprings of vulnerability are risk. In particular, financial risk refers to the possibility of an investment asset value declining due to market factors vacillations. However, the influence of bad news such as political turmoil, wars, or economic crises leads to greater fluctuations in financial asset prices. All

these led to the development of generalized autoregressive conditional heteroscedasticity (GARCH) models for evaluating the risk exposure of financial institutions. Recent studies on estimating volatility focus on modeling financial asset returns using the GARCH models with existing conditional distributions, namely the normal, skew-normal, Student-t, generalized error distribution and their skew versions (Yelamanchili, 2020; Samson et al., 2020a; 2020b). In any case, the previously mentioned broadly utilized conditional innovation distributions have their curious issues and the most remarkable downside is the lack of stability under aggregation, which is of specific significance in financial risk management (Calzolari et al., 2014). More so, the tail strength of the GARCH models remains too short even with the existing conditional distributions which makes it difficult to capture properly the stylized features in financial asset returns (Feng and Shi, 2017). The distributional assumption on the innovation of the GARCH models directly impacts volatility estimates and forecasts. This causes an issue of underestimation and overestimation of true volatility and value at risk (VaR) estimates (Altun et al., 2018; Adubisi and Abdulkadir et al., 2022).

The GARCH models assume that the parameter estimates are non-negative given that it pays more attention to the magnitude of the financial asset returns while disregarding the leverage effect feature in the financial asset returns. Nevertheless, the non-negativity assumption of the model parameters can be violated (Tsay, 2010). To overcome a few shortcomings of the GARCH models, the exponential GARCH (EGARCH) model was proposed to allow for asymmetric effects (Nelson, 1991; Tsay, 2010). A few researchers have studied the performance of the EGARCH models about accurate volatility predictions with skewed and heavy-tailed innovation densities, such as generalized error, beta Student's-t, beta skew Student's-t, Student's-t, and exponentiated generalized Student's-t distribution (Harvey and Chakravarty, 2008; Harvey, 2013; Harvey and Sucarrat, 2013, Agboola et al., 2019; Adubisi and Abdulkadir et al., 2022; Adubisi, Abdulkadir et al., 2022). Chen et al. (2019) proposed the Symmetric Curve (BSC) and Asymmetric Curve Index (ACI) as tools for asymmetric GARCH volatility models in exploring the asymmetric features and depicting time-varying volatility of wind power time series. They concluded that models considering the asymmetric effect of volatility produced good wind power prediction performance. Mohammed et al. (2020) proposed a nonlinear semiparametric fuzzy-EGARCH-ANN model to solve the issue of modeling and prediction of stock market volatility by combining the FIS, ANN, and EGARCH models. The new model was able to capture volatility clustering and leverage the effect of extremely nonlinear and complex financial time series datasets. Adubisi, Abdulkadir et al. (2022) proposed the use of generalized odd generalized exponentiated skew-t density that is heavy-tail and able to capture extreme events than existing innovation densities in some GARCH-type models. Altun et al. (2018; 2019), and Adubisi and Abdulkadir et al. (2022) emphasized that heavy-tailed innovation distributions produce accurate daily volatility and VaR predictions. However, the introduction of new flexible innovation distributions is still vital in increasing the accuracy of the monetary risk estimations.

Additionally, the new direction in distribution theory is the comparison of estimation procedures in estimating the parameters of proposed distributions (see, Chesneau et al., 2020; Ramos et al., 2020; Aldahlan and Afify, 2020; Adubisi and Abdulkadir et al., 2022; Adubisi and Abdulkadir et al., 2022). Motivated by these, the performance of non-Bayesian estimators such as the least squares (LS), maximum likelihood (ML), Weighted least squares (WLS), maximum product spacing (MPS), Cramer-von Mises (CVM), and Anderson-Darling (ANDA) using Monte Carlo experiment is carried-out for the odd exponentiated skew-t (OE_{ST}) distribution. This means, creating a standard guideline for choosing the best estimator for the OE_{ST} distribution which is thought to be of interest to applied-statisticians. Furthermore, the performance evaluation of the GARCH-type model in predicting accurate volatility with well-known heavy-tailed densities can be found in some research works. However, there is still no better flexible leptokurtic and skewed conditional density to quantify financial risks. Thus, our curiosity is stimulated to create an apt flexible innovation conditional density when the underlying conditional density is heavy-tailed but unknown for the EGARCH volatility model. The motivation of this work is to propose a new EGARCH model with new heavy-tailed and skewed density to produce more accurate volatility prediction than other well-known EGARCH models. Herewith, a new odd exponentiated skew-t (OE_{ST}) innovation density is introduced and offers more flexibility for accurate volatility prediction. Likewise, a new dynamic EGARCH-OE_{ST} model is introduced for predicting daily volatility based on the EGARCH volatility model with OE_{ST} innovation density.

The rest of the paper is organized as follows. Section 2, the density, distribution, hazard, and quantile functions of the OE_{ST} distribution are presented. In section 3, the six frequentist estimation methods for estimating the OE_{ST} model parameters are presented. Section 4, Monte Carlo experiments using the six estimation methods for the OE_{ST} model are presented. In Section 5, the EGARCH model is presented with well-known innovation densities while the model selection criteria and prediction performance measures are presented in Section 6. In Section 7, the empirical findings are presented and conclusions are provided in Section 8.

2. The Extended Skew-t Model

The two-parameter model titled the odd exponential skew-t (OE_{ST}) model appropriate for modeling right-skewed, left-skewed, and heavy-tailed datasets introduced by Adubisi et al. (2021) is utilized. The cumulative distribution function (cdf) and probability density function (pdf) are given by:

$$F(y;\varphi,\theta)=\left\{1-e^{-\varphi\frac{H(y;\theta)}{\bar{H}(y;\theta)}}\right\}, \tag{1}$$

and

$$f(y;\varphi,\theta)=\varphi\frac{h(y;\theta)}{\bar{H}(y;\theta)^{2}}e^{-\varphi\frac{H(y;\theta)}{\bar{H}(y;\theta)}}, \tag{2}$$

The hazard rate function (hrf) of the OE_{ST} model is given as

$$h(y)=\frac{\varphi\theta}{2(\theta+y^{2})^{3/2}\left[\bar{H}(y;\theta)\right]^{2}}, \tag{3}$$

where $H(y)=1/2+y\big/2\sqrt{\theta+y^{2}}$, $h(y)=\theta\big/2(\theta+y^{2})^{3/2}$, $\bar{H}(y)=1-H(y)$, $\varphi>0$ is the shape parameter, θ controls the skewness and $y\in\Re$.

Figure 1 depicts that the density function of the OE_{ST} model can be unimodal, symmetric, right and left skewed for several parameter values. This shows that the model could be a better candidate for capturing the financial returns stylized features such as leptokurtic, skewness, and fat-tail thereby increasing the accuracy of volatility predictions.

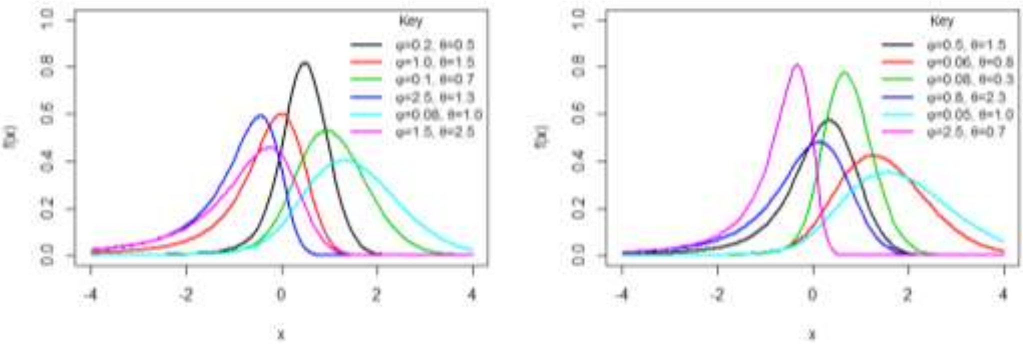

Figure 1. Plots of the OE_{ST} Density Function for Selected Parameter Values
Source: Research finding.

2.1 Quantile Function

The quantile function, $Q(u)=F^{-1}(u)$ is derived by inverting Equation (1). The quantile function of the OE_{ST} model is derived as:

$$Q(u)=\theta^{1/2}\frac{\left[2(\Pi_{u})-1\right]}{\left\{1-\left[2(\Pi_{u})-1\right]^{2}\right\}^{1/2}},\qquad u\in(0,1). \tag{4}$$

where $\Pi_u = \frac{-\log(1-u)}{(\varphi - \log(1-u))}$.

Using the quantile function in Equation (4), various quantile measures such as Bowley's skewness and Moor's kurtosis can be estimated. Figure 2 depicts the 3D plots of the skewness and kurtosis of the OE_{ST} model for some parameter values and evidently, the OE_{ST} model can be used in modeling various datasets with skewed and leptokurtic features. The quantile function in Equation (4) will be used in the Monte Carlo Experiments.

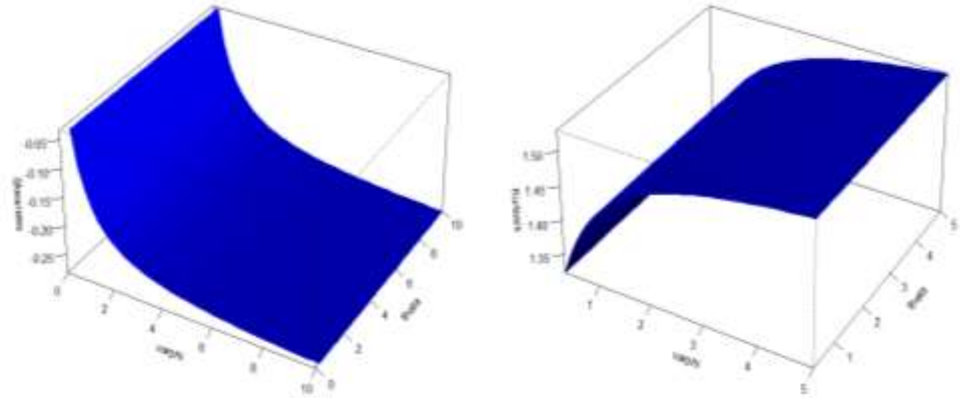

Figure 2. Bowley's Skewness and Moor's Kurtosis Plots of the OE_{ST} Distribution
Source: Research finding.

2.2 Standardized Extended Skew-t Model

The standardized OE_{ST} model is obtained via the transformation $\varepsilon = z\sqrt{h^2}$ where $E(z)=0$ and $\mathrm{var}(z)=1$. The random variable z can be expressed as $z = y - \mu/\sqrt{h^2} = \varepsilon/\sqrt{h^2}$ and $\partial z/\partial \varepsilon = 1/\sqrt{h^2}$. Hence, the standardized OE_{ST} density function takes the form

$$f(z) = \frac{\varphi\theta}{2\left[\theta + z_t^2\right]^{3/2}}\left[1-(\eta)\right]^{-2} \exp\frac{-\varphi(\eta)}{\left[1-(\eta)\right]}\left(1/\sqrt{h^2}\right) \tag{5}$$

where μ and $\sqrt{h^2}$ denotes the mean and standard deviation and $\eta = \left(\frac{1}{2} + \frac{z_t}{2\sqrt{\theta + z_t^2}}\right)$.

The standardized extended skew-t model will be used as the new conditional innovation in the EGARCH volatility model.

2.3 Location-Scale Form of Extended Skew-T Model

The density function of the OE_{ST} model with location parameter μ and scale parameter σ given that $X = \mu + \sigma Y$ takes the form:

$$f(x;\theta,\varphi,\mu,\sigma)=\frac{\varphi\theta}{2\left[\theta+\Psi^2\right]^{3/2}}\left\{1-\left[\frac{1}{2}+\frac{\Psi}{2\sqrt{\theta+\Psi^2}}\right]\right\}^{-2}\exp\frac{-\varphi\left(\frac{1}{2}+\frac{\Psi}{2\sqrt{\theta+\Psi^2}}\right)}{\left\{1-\left[\frac{1}{2}+\frac{\Psi}{2\sqrt{\theta+\Psi^2}}\right]\right\}} \qquad (6)$$

where $\Psi=\left(\frac{x-\mu}{\sigma}\right)$, $\mu\in\Re$ and $\sigma>0$. The density function Equation (6) is denoted as $X\sim OE_{ST}(x;\varphi,\theta,\mu,\sigma)$. This is the location-scale form of the density function given in Equation (2) which is very important in reliability analysis.

3. Estimation Procedures

This section discourses six estimation procedures such as the method of maximum likelihood (ML), the method of maximum product of spacing (MPS), the method of ordinary least squares (OLS), the method of weighted least squares (WLS), the method of Cramer-von Mises (CVM), method of Anderson Darling (ANDA) and method of right-tail Anderson Darling (RANDA) to estimate the OE_{ST} parameters. This is done to find the best estimation method for estimating the parameters of the OE_{ST} for complete samples.

3.1 The ML

The ML is considered in estimating the unknown parameters of the OE_{ST} for complete samples. Let $y_1, y_2, \ldots, y_S$ be the observed values of size (S) from the procedures with parameter vector $\psi=(\varphi,\theta)'$. Hence, the log-likelihood function l takes the form

$$l=S\log\varphi+S\log\theta-S\log 2-3/2\sum_{j=1}^{S}\log\left(\theta+y_j^2\right)-\varphi\sum_{j=1}^{S}\frac{k_j}{\left(1-k_j\right)}-2\sum_{j=1}^{S}\log\left(1-k_j\right) \qquad (7)$$

where $k_j=\left(1/2+y_j\Big/2\sqrt{\theta+y_j^2}\right)$. The ML estimates of the unknown parameters φ and θ of the OE_{ST} can be found by maximizing Equation (7) using the R-software (Optim function), SAS (PROC NLMIXED), or solving the nonlinear likelihood equations found by differentiating the log-likelihood function l. The associated score function $U(\psi)=\left(\frac{\partial l}{\partial\varphi},\frac{\partial l}{\partial\theta}\right)'$, are given as:

$$U_\varphi(\psi)=\frac{S}{\varphi}-\sum_{j=1}^{S}\frac{k_j}{\left(1-k_j\right)}, \text{ and}$$

$$U_\theta(\psi)=\frac{S}{\theta}-\frac{3}{2}\sum_{j=1}^{S}\frac{1}{p_j}+\varphi\sum_{j=1}^{S}\frac{y_j}{\sqrt{p_j}\left(\sqrt{p_j}-y_j\right)^2}+\sum_{j=1}^{S}\frac{y_j}{p_j\left(-\sqrt{p_j}+y_j\right)},$$

where $p_j = \theta + y_j^2$.

The Newton-Rapshon method can be utilized in solving the nonlinear equations system. For simplicity, from $U_\varphi(\psi) = \partial l/\partial\varphi = 0$ for fixed θ. The solution for $\hat{\varphi}$ can be expressed as

$$\hat{\varphi} = \frac{S}{-\sum_{j=1}^{S} \frac{k_j}{\left(1-k_j\right)}} \tag{8}$$

The ML estimates of φ can be calculated from Equation (8) as $\hat{\varphi}$ while statistical softwares such as R-software can be used in solving for the ML estimates of θ denoted by $\hat{\theta}$ numerically via iterative methods from Equation (9).

$$U_\theta(\psi) = \frac{S}{\theta} - \frac{3}{2}\sum_{j=1}^{S}\frac{1}{p_j} + \varphi\sum_{j=1}^{S}\frac{y_j}{\sqrt{p_j}\left(\sqrt{p_j} - y_j\right)^2} + \sum_{j=1}^{S}\frac{y_j}{p_j\left(-\sqrt{p_j} + y_j\right)} = 0 \tag{9}$$

3.2 The OLS and WLS

Let $y_{(1:S)}, y_{(2:S)}, \ldots, y_{(S:S)}$ be the ordered sample of size (s) from Equation (1) of the OE_{ST}. The OLS $\hat{\varphi}_{OLS}$ and $\hat{\theta}_{OLS}$ can be found by minimizing:

$$OL(\varphi,\theta) = \sum_{j=1}^{S}\left[F\left(y_{(j)}\middle|\varphi,\theta\right) - \xi(j,S)\right]^2$$

where $\xi(j,S) = j/(S+1)$, concerning φ and θ or equivalently obtained by solving the following differential equation

$$\sum_{j=1}^{S}\left[F\left(y_{(j)}\middle|\varphi,\theta\right) - \xi(j,S)\right]\Delta_i\left(y_{(j)}\middle|\varphi,\theta\right) = 0,\ i = 1,2\,,$$

where

$$\Delta_1\left(y_{(j)}\middle|\varphi,\theta\right) = \frac{\partial}{\partial\varphi}F\left(y_{(j)}\middle|\varphi,\theta\right) \text{ and } \Delta_2\left(y_{(j)}\middle|\varphi,\theta\right) = \frac{\partial}{\partial\theta}F\left(y_{(j)}\middle|\varphi,\theta\right) \tag{10}$$

The solutions of Δ_i for $i = 1,2$ can be found numerically. For more details, see Swain et al. (1988).

Similarly, the WLS $\hat{\varphi}_{WLS}$ and $\hat{\theta}_{WLS}$ can be found by minimizing:

$$WL(\varphi,\theta) = \sum_{j=1}^{S}\Phi(j,S)\sum_{j=1}^{S}\left[F\left(y_{(j)}\middle|\varphi,\theta\right) - \xi(j,S)\right]^2,$$

where $\Phi(j,S) = (S+1)^2(S+2)/j(S-j+1)$, relative to φ and θ or can be obtained by solving the following differential equation

$$\sum_{j=1}^{S}\Phi(j,S)\left[F\left(y_{(j)}\middle|\varphi,\theta\right) - \xi(j,S)\right]\Delta_i\left(y_{(j)}\middle|\varphi,\theta\right) = 0,\ i = 1,2\,,$$

where $\Delta_1(.|\varphi,\theta)$ and $\Delta_2(.|\varphi,\theta)$ are given in Equation (10).

3.3 The MPS

The MPS estimator for the estimation of unknown parameters with an ordered sample $y_{(1:S)}, y_{(2:S)}, \ldots, y_{(S:S)}$ from Equation (1) of the OE$_{ST}$ and uniform spacing for this random sample, proposed by Cheng and Amin (1979, 1983) is given by $D_j(\varphi,\theta) = F\left(y_{(j:S)}|\varphi,\theta\right) - F\left(y_{(j-1:S)}|\varphi,\theta\right)$, for $j = 1,2,\ldots,S+1$, where $F\left(y_{(0:S)}|\varphi,\theta\right) = 0$, $F\left(y_{(S+1:S)}|\varphi,\theta\right) = 1$ and $\sum_{j=1}^{S+1} D_j(\varphi,\theta) = 1$. The MPS $\hat{\varphi}_{MPS}$ and $\hat{\theta}_{MPS}$ can be obtained by maximizing the geometric mean (GM) of the spacing:

$$GM(\varphi,\theta) = \left[\prod_{j=1}^{S+1} D_j(\varphi,\theta)\right]^{1/S+1} \tag{11}$$

relative to φ and θ or by maximizing the logarithm of GM of the spacing:

$$LGM(\varphi,\theta) = \frac{1}{S+1}\sum_{j=1}^{S+1} \log D_j(\varphi,\theta),$$

The MPS $\hat{\varphi}_{MPS}$ and $\hat{\theta}_{MPS}$ of the OE$_{ST}$ can be obtained by solving the differential equation $\sum_{j=1}^{S+1} \frac{1}{D_j(\varphi,\theta)}\left[\Delta_i\left(y_{(j:S)}|\varphi,\theta\right) - \Delta_i\left(y_{(j-1:S)}|\varphi,\theta\right)\right] = 0,\ i = 1,2$,

where $\Delta_1(.|\varphi,\theta)$ and $\Delta_2(.|\varphi,\theta)$ are given in Equation (10).

3.4 The ANDA

The ANDA $\hat{\varphi}_{ANDA}$ and $\hat{\theta}_{ANDA}$ (Anderson and Darling, 1952) can be obtained for the OE$_{ST}$ by minimizing the function given by

$$AD(\varphi,\theta) = -S - \frac{1}{S}\sum_{j=1}^{S}(2j-1)\left[\log F\left(y_{(j:S)}|\varphi,\theta\right) + \log \bar{F}\left(y_{(S+1-j:S)}|\varphi,\theta\right)\right] \tag{12}$$

relative to φ and θ. Likewise, the estimates can be obtained by solving the nonlinear equation $\sum_{j=1}^{S}(2j-1)\left[\frac{\Delta_i\left(y_{(j:S)}|\varphi,\theta\right)}{F\left(y_{(j:S)}|\varphi,\theta\right)} - \frac{\Delta_k\left(y_{(S+1-j:S)}|\varphi,\theta\right)}{\bar{F}\left(y_{(S+1-j:S)}|\varphi,\theta\right)}\right] = 0,\ i,k = 1,2$,

where $\Delta_1(.|\varphi,\theta)$ and $\Delta_2(.|\varphi,\theta)$ are given in Equation (10).

3.5 The CVM

The CVM $\hat{\varphi}_{CVW}$ and $\hat{\theta}_{CVW}$ (MacDonald, 1971) for the OE$_{ST}$ parameters are obtained by minimizing the function given by:

$$CV(\varphi,\theta) = \frac{1}{12S} + \sum_{j=1}^{S}\left[F\left(y_{(j:S)}|\varphi,\theta\right) - \frac{2(j-1)+1}{2S}\right]^2 \tag{13}$$

relative to φ and θ. Solving the nonlinear equation, the CVM estimates can also be obtained: $\sum_{j=1}^{S}\left[F\left(y_{(j:S)}|\varphi,\theta\right) - \frac{2(j-1)+1}{2S}\right]\Delta_i(y_{(j:S)}|\varphi,\theta) = 0,\ i = 1,2$

where $\Delta_1(.|\varphi,\theta)$ and $\Delta_2(.|\varphi,\theta)$ are given in Equation (10).

4. EGARCH Volatility Model

The autoregressive conditional heteroscedasticity (ARCH) model was introduced by Engle (1982) for modeling time-varying volatility. Bollerslev (1986) proposed the symmetric generalized ARCH model (GARCH) to cater to the shortfalls of the ARCH model. However, the issue of leverage effect has been in the discussion for so long as recorded in the literature.

In this research, the asymmetric exponential GARCH (EGARCH) model introduced by Nelson (1991) is considered. The EGARCH differs from the symmetric GARCH variance structure given that the natural log variance is used which suggests that parameters are unrestricted, that is, the parameters are allowed to take negative values while ensuring a positive conditional variance. Moreso, the EGARCH specification includes the asymmetric impact of positive and negative shocks on volatility. The log-return of daily assets is symbolized as r_t. The EGARCH (1,1) model is defined as:

$$\begin{aligned} &r_t = \mu + \varepsilon_t, \\ &\varepsilon_t = z_t\sqrt{h_t^2}, \quad z_t \sim i.i.d. \\ &\ln h_t^2 = \gamma_0 + \gamma_1 \frac{\varepsilon_{t-1}}{\sqrt{h_{t-1}^2}} + \gamma_3 \left| \frac{\varepsilon_{t-1}}{\sqrt{h_{t-1}^2}} \right| + \gamma_2 \ln h_{t-1}^2, \end{aligned} \tag{14}$$

where $\gamma_0 > 0$, $\gamma_1 > 0$, $\gamma_2 > 0$. Here, z_t is the conditional innovation density with $\mathrm{E}(z_t) = 0$ and $\mathrm{var}(z_t) = 1$, μ_t is the conditional mean, γ_3 is the leverage parameter, $\ln(h_t^2)$ is conditional log variance at the present day t, ε_{t-1} and $\ln(h_{t-1}^2)$ are the error and conditional log variance on the preceding day $t-1$, respectively. In asymmetric volatility models, negative shocks have a larger effect on volatility than positive shocks when the parameter γ_3 is positive. The accuracy of volatility predictions is directly affected by the innovation process distributional assumption (Altun et al., 2018). The distributional assumption on the innovation directly impacts on the EGARCH models estimates and forecasts. These conditional innovation distributions have their curious issues which the most remarkable downside is the lack of stability under aggregation (Calzolari et al., 2014). Additionally, the tail strength of the EGARCH models remains too short even with the existing conditional densities which makes it difficult to capture properly the stylized features in financial asset returns (Feng and Shi, 2017). This causes an issue of underestimation and overestimation of true returns volatility. The commonly existing conditional innovation densities are provided in the Appendix.

4.1 New Conditional Innovation Density

For the standardized OE_{ST} distribution introduced in subsection 2.2, the log-likelihood function takes the form

$$l(\xi) = T\ln\varphi + T\ln\theta - T\ln 2 - 3/2\sum_{t=1}^{T}\ln\left[\theta + z_t^2\right] - \varphi\sum_{t=1}^{T}\frac{\eta_t}{1-\eta_t} - 2\sum_{t=1}^{T}\ln\left(1-\eta_t\right) - 0.5\sum_{t=1}^{T}\ln\left(h_t^2\right) \quad (15)$$

where $\xi = (\mu, \gamma_0, \gamma_1, \gamma_2, \gamma_3, \varphi, \theta)$ denotes the parameter vector of the EGARCH-Odd exponentiated skew-t (EGARCH-OE_{ST}), θ is the skewness parameter, φ is the shape parameter and $\eta_t = \left(\frac{1}{2} + \frac{z_t}{2\sqrt{\theta + z_t^2}}\right)$.

5. Evaluation of Volatility Models

5.1 Model Selection Criteria

The selection of the most appropriate EGARCH model for modeling and forecasting financial data sets is done using two information criteria. The modified Akaike information criteria (AIC) and Bayesian information criteria (BIC) proposed by Brooks and Burke (2003) are utilized in selecting the best model under the conditional innovation densities. The modified AIC and BIC criteria are given by:

$$AIC = \frac{2k}{T} - \frac{2LL}{T}$$
$$BIC = \frac{k\log_e(T)}{T} - \frac{2LL}{T} \quad (16)$$

where k is the total number of estimated parameters, the estimated log-likelihood value is denoted by LL and T is the sample size. The EGARCH model with the least AIC and BIC values is regarded as the most appropriate model under the specified conditional innovation density.

5.2 Forecasts Performance

The forecasts performance of the EGARCH models is appraised using the mean square error (MSE), root mean square root (RMSE), and mean absolute error (MAE). The performance measures for the volatility forecasts are given by:

$$MSE = \frac{1}{T}\sum_{t=1}^{T}\left(\hat{h}_t - h_t\right)^2$$
$$RMSE = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(\hat{h}_t - h_t\right)^2} \quad (17)$$
$$MAE = \frac{1}{T}\sum_{t=1}^{T}\left|\hat{h}_t - h_t\right|$$

where $\hat{h}_t$ and represents the volatility forecast and realized volatility, and T is the sample size. The model with the least performance measures is regarded as the most appropriate for predicting the volatility of the daily log-returns.

5.3 Monte Carlo Experiments

The Monte Carlo experiments are provided to compare the performance of the six methods discussed in Section 3. Three different parameter combinations (Comb) are considered: Comb 1 $(\varphi = 0.8, \theta = 1.5)$, Comb 2 $(\varphi = 1.5, \theta = 2.0)$, Comb 3 $(\varphi = 2.0, \theta = 0.8)$. The datasets are generated from the OE_{ST} model under these combinations by selecting S = 20, 75, 150, and 250. For each combination, $N = 3000$ Pseudo-random samples are generated from the inverse cdf of the OE_{ST} model. The experiments are executed in an R-environment and the average values (AVEs), average absolute-biases (AABs), and root-mean-square errors (RMSEs) of the parameter (Pa.) estimates are calculated with:

$$\begin{aligned} AVE(\hat{\vartheta}_i) &= \frac{1}{3000}\sum_{j=1}^{3000}\hat{\vartheta}_{i,j} \\ AAB(\hat{\vartheta}_i) &= \frac{1}{3000}\sum_{j=1}^{3000}\left|\hat{\vartheta}_{i,j} - \vartheta_i\right| \\ RMSE(\hat{\vartheta}_i) &= \sqrt{\frac{1}{3000}\sum_{j=1}^{3000}\left(\hat{\vartheta}_{i,j} - \vartheta_i\right)^2}, \end{aligned} \tag{18}$$

where $\vartheta = (\varphi, \theta)$ and $\hat{\vartheta} = \left(\hat{\varphi}, \hat{\theta}\right)$. The AVE, AAB, and RMSE values are provided in Tables 1 to 3 and Figures 3 to 8 depict the AAB and RMSE plots. The following conclusions are reached:

- The estimators are asymptotically unbiased given that their biases converge to zero as the sample size increases.
- The estimators are consistent given that their RMSE tends to zero for large sample sizes.
- The ML and MPS perform better than the other estimators in terms of minimum biases and RMSE in most cases.
- The LSQ has the largest biases and RMSEs compared to other estimators in most cases.
- Therefore, the unknown parameters of the OE_{ST} model can be best estimated using either the ML or MPS methods.

Table 1. Monte Carlo Experiment Results of the Estimators for $(\varphi = 0.8, \theta = 1.5)$

S	Measures	Pa.	ML	MPS	ANDA	CVM	OLS	WLS
20	AVE	φ	0.8289	0.8176	0.8252	0.8386	0.8264	0.8249
		θ	1.4833	1.9236	1.6377	1.5550	1.8107	1.7394
	AAB	φ	0.0289	0.0176	0.0252	0.0386	0.0264	0.0249
		θ	0.0167	0.4236	0.1377	0.0550	0.3107	0.2394
	RMSE	φ	0.2262	0.1990	0.2174	0.2438	0.2167	0.2169
		θ	0.6701	0.9709	0.7810	0.7685	0.9461	0.8642
75	AVE	φ	0.8058	0.8021	0.8058	0.8077	0.8063	0.8056
		θ	1.4938	1.6325	1.5321	1.5043	1.5694	1.5382
	AAB	φ	0.0058	0.0021	0.0058	0.0077	0.0063	0.0056
		θ	0.0062	0.1325	0.0321	0.0043	0.0694	0.0382
	RMSE	φ	0.1099	0.1055	0.1100	0.1145	0.1118	0.1102
		θ	0.3356	0.3908	0.3611	0.3841	0.4052	0.3674
150	AVE	φ	0.8011	0.7988	0.8012	0.8021	0.8015	0.8010
		θ	1.4906	1.5671	1.5093	1.4972	1.5292	1.5095
	AAB	φ	0.0011	0.0012	0.0012	0.0021	0.0015	0.0010
		θ	0.0094	0.0671	0.0093	0.0028	0.0292	0.0096
	RMSE	φ	0.0765	0.0748	0.0769	0.0793	0.0784	0.0769
		θ	0.2331	0.2541	0.2464	0.2653	0.2720	0.2485
250	AVE	φ	0.8008	0.7992	0.8008	0.8014	0.8010	0.8007
		θ	1.4927	1.5420	1.5038	1.4964	1.5155	1.5029
	AAB	φ	0.0008	0.0008	0.0008	0.0014	0.0010	0.0007
		θ	0.0073	0.0420	0.0038	0.0036	0.0155	0.0029
	RMSE	φ	0.0595	0.0586	0.0600	0.0619	0.0614	0.0601
		θ	0.1813	0.1921	0.1905	0.2053	0.2083	0.1911

Source: Research finding.

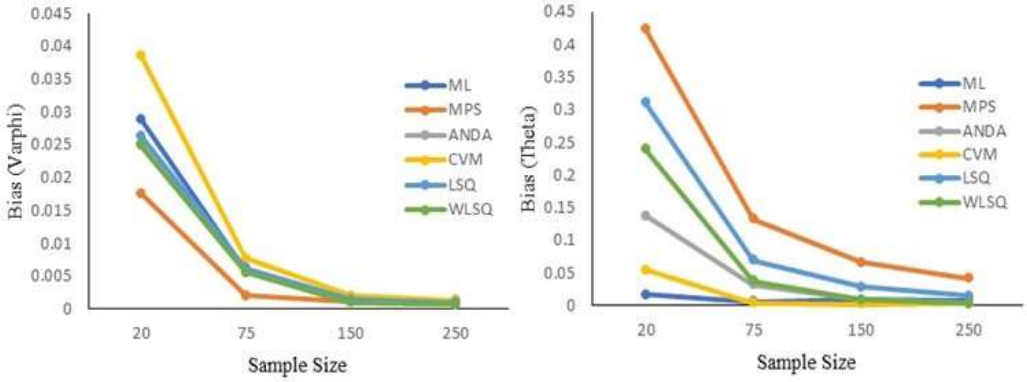

Figure 3. AAB for the Estimates of the Six Estimation Methods
Source: Research finding.

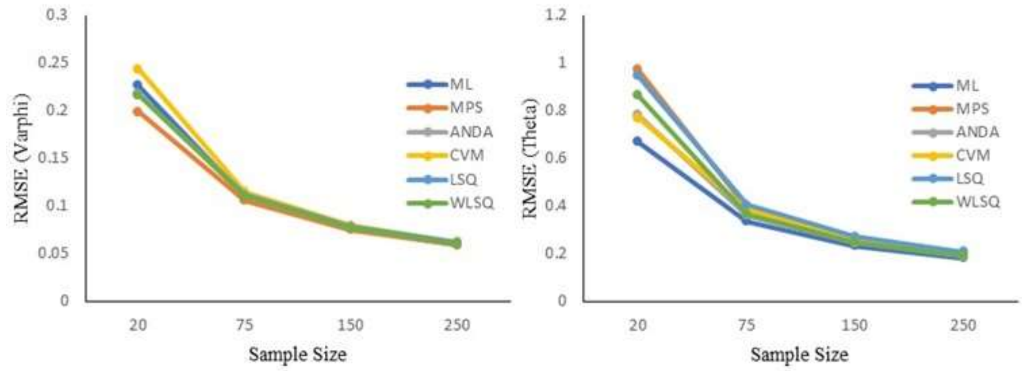

Figure 4. RMSE for the Estimates of the Six Estimation Methods
Source: Research finding.

Table 2. Monte Carlo Experiment Results of the Estimators for $(\varphi = 1.5, \theta = 2.0)$

S	Measures	Pa.	ML	MPS	ANDA	CVM	OLS	WLS
20	AVE	α	1.6262	1.4890	1.5717	1.6572	1.5864	1.5646
		φ	1.9539	2.5850	2.2206	2.1140	2.4895	2.3709
	AAB	α	0.1262	0.0111	0.0717	0.1572	0.0864	0.0646
		φ	0.0461	0.5848	0.2206	0.1140	0.4895	0.3709
	RMSE	α	0.4844	0.3632	0.4277	0.7380	1.6801	0.7633
		φ	0.9335	1.3681	1.1599	1.1697	1.4475	1.2874
75	AVE	α	1.5251	1.4820	1.5158	1.5350	1.5117	1.5150
		φ	1.9872	2.1920	2.0531	2.0126	2.1075	2.0588
	AAB	α	0.0251	0.0175	0.0158	0.0350	0.0117	0.0150
		φ	0.0128	0.1915	0.0513	0.0126	0.1075	0.0588
	RMSE	α	0.1882	0.1745	0.1906	0.2204	0.2084	0.1958

		φ	0.4581	0.5406	0.5145	0.5611	0.5942	0.5244
150	AVE	α	1.5107	1.4860	1.5065	1.5149	1.5038	1.5064
		φ	1.9855	2.1000	2.0169	1.9994	2.0458	2.0157
	AAB	α	0.0107	0.0136	0.0065	0.0149	0.0038	0.0064
		φ	0.0145	0.1005	0.0169	0.0006	0.0458	0.0157
	RMSE	α	0.1294	0.1246	0.1324	0.1460	0.1423	0.1338
		φ	0.3191	0.3519	0.3496	0.3860	0.3966	0.3532
250	AVE	α	1.5069	1.4910	1.5046	1.5096	1.5030	1.5047
		φ	1.9892	2.0650	2.0075	1.9967	2.0244	2.0053
	AAB	α	0.0069	0.0092	0.0046	0.0096	0.0030	0.0047
		φ	0.0108	0.0649	0.0075	0.0033	0.0244	0.0053
	RMSE	α	0.0993	0.0971	0.1025	0.1122	0.1150	0.1035
		φ	0.2474	0.2645	0.2697	0.2982	0.3028	0.2708

Source: Research finding.

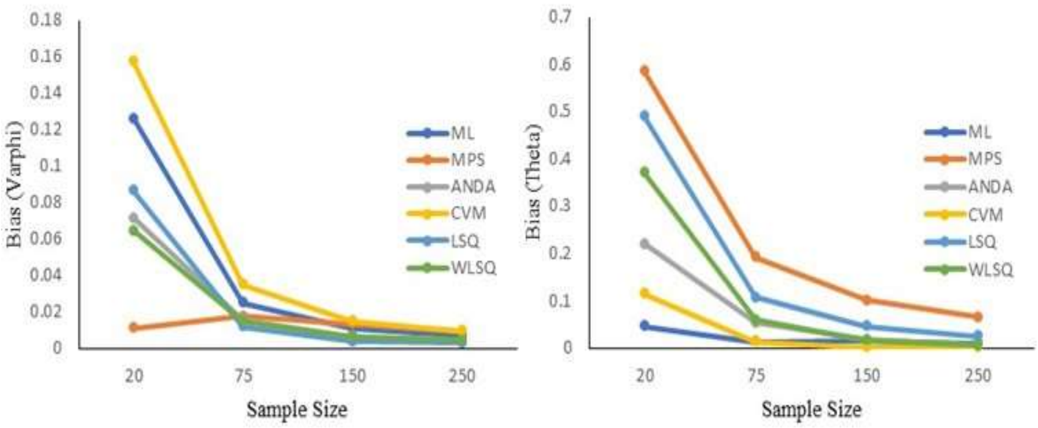

Figure 5. AAB for the Estimates of $\varphi = 1.5, \theta = 2.0$ for the Six Estimation Methods

Source: Research finding.

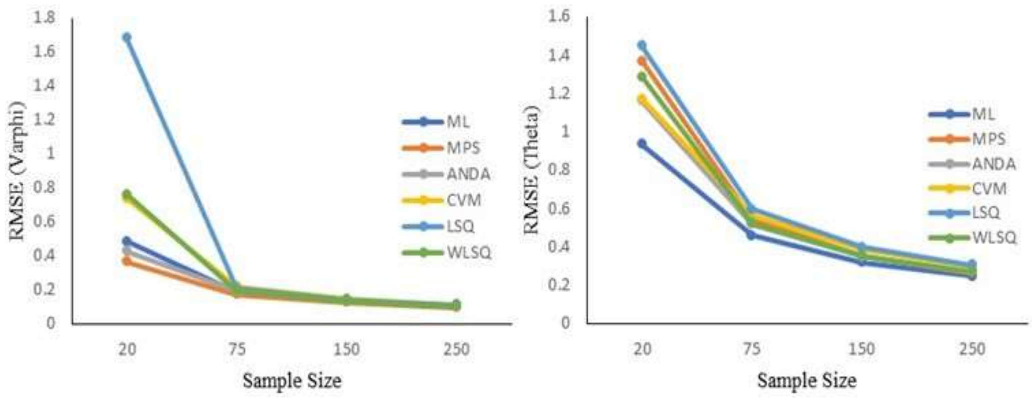

Figure 6. RMSE for the Estimates of $\varphi = 1.5, \theta = 2.0$ for the Six Estimation Methods
Source: Research finding.

Table 3. Monte Carlo Experiment Results of the Estimators for $(\varphi = 2.0, \theta = 0.8)$

S	Measures	Pa.	ML	MPS	ANDA	CVM	OLS	WLS
20	AVE	α	2.2760	1.9790	2.1392	2.2919	2.0694	2.1000
		φ	0.7744	1.0470	0.9004	0.8643	1.0272	0.9692
	AAB	α	0.2760	0.0209	0.1392	0.2919	0.0694	0.1000
		φ	0.0256	0.2473	0.1004	0.0643	0.2272	0.1692
	RMSE	α	0.9858	0.5850	0.7146	1.3641	0.8095	0.8908
		φ	0.3962	0.5839	0.5101	0.5200	0.6511	0.5662
75	AVE	α	2.0485	1.9644	2.0281	2.0702	2.0230	2.0287
		φ	0.7931	0.8823	0.8245	0.8067	0.8481	0.8262
	AAB	α	0.0485	0.0356	0.0281	0.0702	0.0230	0.0287
		φ	0.0069	0.0823	0.0245	0.0067	0.0481	0.0262
	RMSE	α	0.2667	0.2409	0.2736	0.3488	0.3220	0.2895
		φ	0.1922	0.2283	0.2212	0.2449	0.2598	0.2257
150	AVE	α	2.0224	1.9740	2.0134	2.0313	2.0091	2.0141
		φ	0.7933	0.8440	0.8081	0.8004	0.8206	0.8071
	AAB	α	0.0224	0.0255	0.0134	0.0313	0.0091	0.0141
		φ	0.0067	0.0440	0.0081	0.0004	0.0206	0.0071
	RMSE	α	0.1799	0.1712	0.1881	0.2215	0.2134	0.1928
		φ	0.1338	0.1484	0.1498	0.1681	0.1729	0.1515
250	AVE	α	2.0142	1.9827	2.0093	2.0199	2.0067	2.0101
		φ	0.7951	0.8285	0.8037	0.7990	0.8110	0.8025
	AAB	α	0.0142	0.0173	0.0093	0.0199	0.0067	0.0101

	φ	0.0049	0.0285	0.0037	0.0010	0.0110	0.0025
RMSE	α	0.1362	0.1319	0.1444	0.1670	0.1633	0.1469
	φ	0.1036	0.1114	0.1154	0.1297	0.1318	0.1159

Source: Research finding.

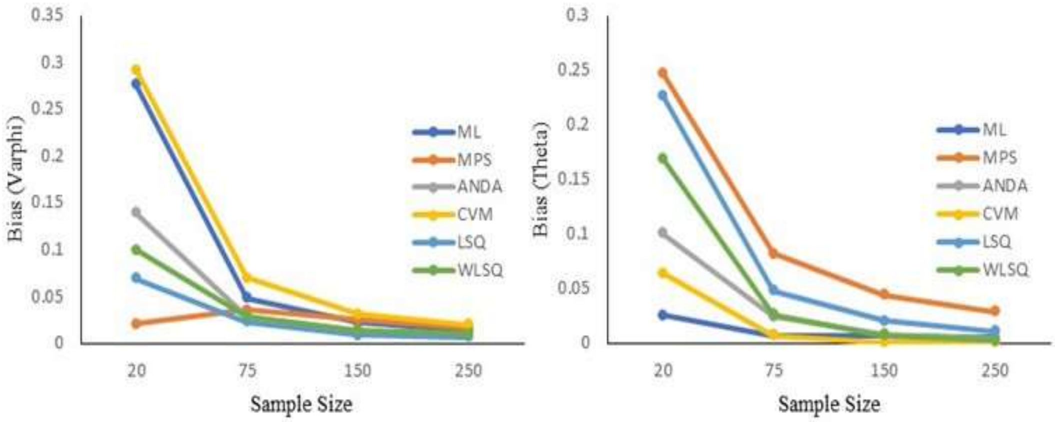

Figure 7. AAB for the Estimates of $\varphi = 2.0, \theta = 0.8$ for the Six Estimation Methods

Source: Research finding.

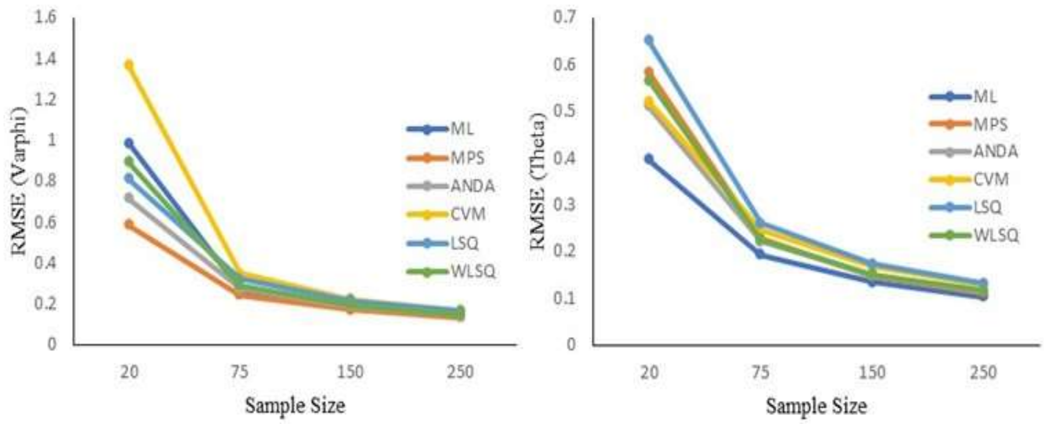

Figure 8. RMSE for the Estimates of $\varphi = 2.0, \theta = 0.8$ for the Six Estimation Methods

Source: Research finding.

6. Empirical Finding

6.1 Data Report

To appraise the performance of the EGARCH models in forecasting daily volatility, the First Bank Nigeria (FBN) stock price is used. The utilized dataset consists of 2304 daily log-returns from 31/01/2012 to 31/05/2021. The estimation process is carried out using 2203 daily log-returns and 101 daily log-returns are used for forecasting (out-of-sample) performance assessment of the models. The summary statistics of the daily log-returns of the FBN index for the estimation and prediction processes are provided in Table 4. Figure 9 displays the daily log-returns of the FBN index and histogram of log-returns.

Table 1 shows positive skewness and high excess kurtosis, leading to a large Jarque-Bera (JB) statistic ($p < 0.001$) signifying that the daily log-returns for the estimation process have non-normality characteristics. Further, the ARCH Lagrange-multiplier (LM) and Ljung Box-Q tests at lag 10, indicate the incidence of conditional heteroscedasticity and autocorrelation in the FBN log-returns.

Table 4. Summary Statistics for the FBN Daily Log-Returns

Estimation process					
Number of observations	Mean	Median	Minimum	Maximum	Std Dev.
2203	-0.015	0.000	-10.536	9.758	2.747
Skewness	Kurtosis	Jarque-Bera	ARCH (10)	Q (10)	
0.178	2.301	499.84 (p< 0.0001)	187.01 (p < 0.0001)	362.54(p < 0.0001)	
Prediction process					
Number of observations	Mean	Median	Minimum	Maximum	Std Dev.
101	7.304	7.300	6.900	7.850	0.158
Skewness	Kurtosis				
0.430	1.029				

Source: Research finding.

7. Estimation of EGARCH Models Parameters

The EGARCH (1,1) model defined in Eq (15) is estimated under nine different innovation densities: normal (NORM), student-t (ST), generalized error (GE), skew-normal (SNORM), skew student-t (SST), skew generalized error (SGE), generalized hyperbolic (GHYB) and Johnson (SU) reparametrized (JSU). Table 5 provides the estimated parameters of the EGARCH models. The *rugarch* package in the R language of programming is used in estimating the parameters of the EGARCH-NORM, EGARCH-ST, EGARCH-GE, EGARCH-SNORM, EGARCH-SST, EGARCH-SGE, EGARCH-GHB and EGARCH-JSU while the *Optim* function in R-software is utilized to maximize the log-likelihood function of EGARCH-OE_{ST}.

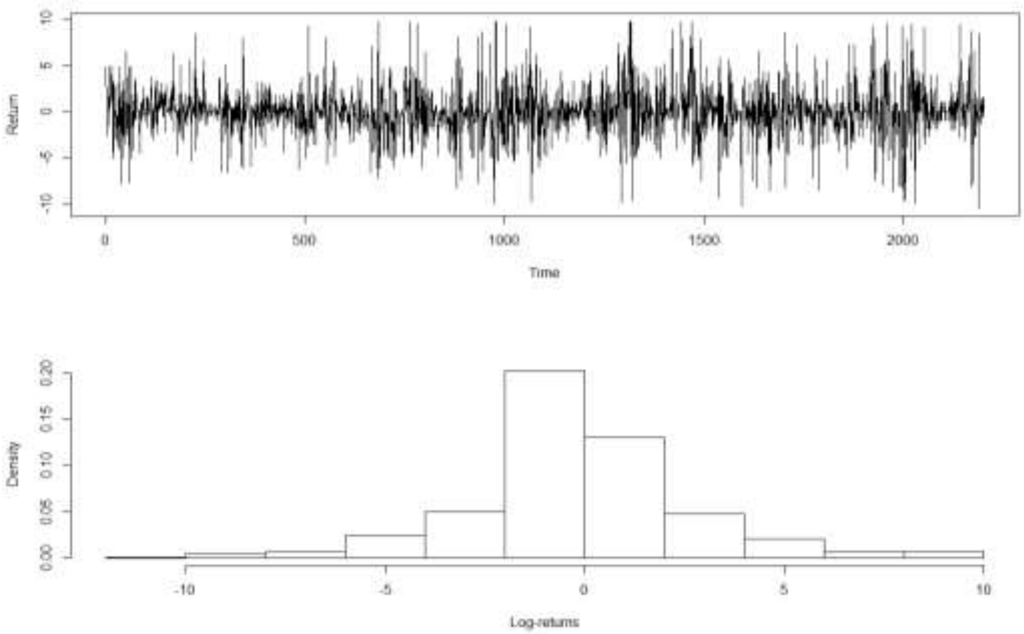

Figure 9. The Daily Log-Returns of the FBN Stock Price (**Top Panel**) and Histogram of Daily Log-Returns (**Bottom Panel**)
Source: Research finding.

As observed in Table 5, the EGARCH-OE_{ST} model has the highest log-likelihood (LL) value and exhibits a greater fit to the standardized residuals compared to others. The parameter estimates of the conditional variance are highly statistically significant and is significant at a standard level which shows that the daily log-returns have a leverage effect. Hence, the impact of the shocks is asymmetric which implies that the impact of negative shocks on volatility is higher than positive shocks of the same size. Some model selection statistic values that show that the EGARCH-OE_{ST} has the least AIC and BIC values relative to others are also provided in Table 5. These results reveal that the EGARCH-OE_{ST} model is best for modeling the FBN log-returns.

Table 5. Parameter Estimates of the EGARCH (1,1) for FBN Log-Returns Assuming Nine Different Innovation Densities

Par	**NORM**	**SNORM**	**ST**	**SST**	**GE**	**SGE**	**GHB**	**JSU**	**OE$_{ST}$**
μ	-0.0625‘***’	-0.0526	-0.0623‘***’	-0.0537	0.0645‘***’	0.1905‘***’	-0.0469	-0.0522	3.886E-08‘*’
γ_0	0.3434‘***’	0.3407‘***’	0.3019	0.8333‘***’	0.2828‘***’	1.3161‘***’	0.2770‘***’	7.5206	3.725E-08‘*’
γ_1	0.0068	0.0059	-0.0048	-0.0368	0.0147	-9.1048‘***’	-0.0043	-0.0891	0.1477‘*’
γ_2	0.8356‘***’	0.8370‘***’	0.8583‘***’	0.8179‘***’	0.8506‘***’	0.8521‘***’	0.8493‘***’	0.8996	0.7576‘***’
γ_3	0.4118‘***’	0.4087‘***’	0.6646	2.1670‘***’	0.5702‘***’	-7.4675‘***’	0.5615	0.1635	0.3715‘*’
α	-	1.0138‘***’	-	-	-	-	0.2500‘*’	-	-
ξ	-	-	3.1364‘***’	2.0656‘***’	0.8723‘***’	-	-	-	-
ν	-	-	-	1.0093‘***’	-	0.1005‘***’	-	-	-
η	-	-	-	-	-	1.0002‘***’	-	-	-
β	-	-	-	-	-	-	0.0009	-	-
λ	-	-	-	-	-	-	0.6316‘***’	-	-
υ	-	-	-	-	-	-	-	-0.0000	-
θ	-	-	-	-	-	-	-	0.1703	1.5878‘*’
φ	-	-	-	-	-	-	-	-	1.1473‘***’
LL	-5156	-5156	-5002	-5019	-4963	-6374	-4971	-7738	**-4445**
AIC	4.6854	4.6861	4.5462	4.5627	4.5113	5.7931	4.5203	7.0317	**4.0420**
BIC	4.6983	4.7016	4.5617	4.5808	4.5268	5.8112	4.5410	7.0498	**4.0612**

Source: Research finding.

Note: Significance levels: < 0.05’***’, < 1’*’.

Table 6 provides the diagnostic test results for the EGARCH model under the various innovation densities used in this study. As seen from Table 6, the Ljung Box-Q statistic ($p > 0.05$) indicates that the squared standardized residuals from the EGARCH-OE_{ST} model exhibit no sign of autocorrelation. Also, the ARCH-LM statistic ($p > 0.05$) indicates that the standardized residuals from the EGARCH-OE_{ST} model exhibit no additional conditional heteroscedasticity, that is, the conditional variance equation is specified correctly. Thus, the results reveal that standardized OE_{ST} distribution provides a better fit to the standardized residuals of the EGARCH (1,1) model.

Table 6. Estimated EGARCH Models Diagnostic Tests

Model	Ljung-Box Statistic	p-value	ARCH-LM Statistic	p-value
EGARCH- NORM	8.400	0.900	0.310	0.600
EGARCH- SNORM	8.356	0.909	0.353	0.552
EGARCH- ST	19.000	0.200	1.400	0.200
EGARCH- SST	20.499	0.148	3.029	0.082
EGARCH- GE	16.000	0.400	0.920	0.300
EGARCH-SGE	0.001	1.000	0.001	0.983
EGARCH-GHB	16.359	0.359	0.907	0.359
EGARCH-JSU	2.16E-7	1.000	2202	2.2E-16
EGARCH- OE_{ST}	11.526	0.714	10.877	0.367

Source: Research finding.

8. Forecasts Evaluation of the GARCH-type Models

The evaluation metrics of the EGARCH-$OE_{ST,}$ and other models for the out-of-sample prediction are provided in Table 7. The evaluation metrics indicate that the EGARCH-OE_{ST} has the least MSE, RMSE and MAE values compared to other models under the various innovation densities. Hence, the EGARCH-OE_{ST} model is statistically efficient and displays superior ability in predicting the FBN volatility relative to other models.

Table 7. Forecasts Evaluation Metrics of the Estimated EGARCH Models

Model	MSE	RMSE	MAE
EGARCH- NORM	4.298	2.073	1.200
EGARCH- SNORM	4.297	2.073	1.196
EGARCH- ST	4.298	2.073	1.200
EGARCH- SST	4.297	2.073	1.196
EGARCH- GE	4.298	2.073	1.195
EGARCH-SGE	4.296	2.073	1.194
EGARCH-GHB	4.296	2.073	1.194
EGARCH-JSU	4.297	2.073	1.196
EGARCH- OE_{ST}	**3.838**	**1.959**	**1.125**

Source: Research finding.

9. Conclusion

The estimation of the odd exponentiated skew-t (OE_{ST}) model parameters using the maximum product of spacing, Anderson-Darling, right Anderson Darling, maximum likelihood, Cramer-von Mises, least squares, and weighted least squares estimation procedures are considered in this study. The density and cumulative functions, failure rate function, quantile function, standardized density function, location-scale density, and the mathematical expressions of the estimators concerning the OE_{ST} model are provided. However, it is practically indeterminate to compare these estimators theoretically, hence Monte Carlo experiments are carried out to study the performance of the estimators. The Monte Carlo results indicates that the estimators are asymptotically unbiased and consistent given that their biases converge to zero as the sample size increases. The MLE and MPS having more advantage with least RMSE values than the other estimators.

Furthermore, the standardized odd exponentiated skew-t (OE_{ST}) density is introduced as a new distributional innovation assumption for the EGARCH volatility model. The modeling of the volatility of FBN log-returns with the EGARCH (1,1) under OE_{ST} innovation density relative to eight existing innovation densities is carried-out. The empirical findings confirm that based on the log-likelihood, AIC and BIC, the EGRACH-OE_{ST} model is optimally the best model. Likewise, the EGRACH-OE_{ST} model has the least forecast performance measures among other models, hence the standardized OE_{ST} distribution provides better fit to the standardized residuals of the EGARCH (1,1) model. In conclusion, the EGARCH-OE_{ST} model has better in- and -out of-samples performance than other models for the FBN log-returns.

Conflict of Interest

We the authors declare that there is no conflict of interest in this work.

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Appendix

Frequently Utilized Innovation Densities in Garch-Type Models

Normal Distribution

For the standardized normal distribution, the log-likelihood function of r_t takes the form:

$$l(\xi) = -0.5\left(T\ln 2\pi + \sum_{t=1}^{T} z_t^2 + \sum_{t=1}^{T}\ln\left(h_t^2\right)\right)$$

where $\xi = (\mu, \gamma_0, \gamma_1, \gamma_2, \gamma_3)$ denotes the parameter vector of the EGARCH-normal (EGARCH-N) model.

Student-t Distribution

For the standardized student-t distribution, the log-likelihood function takes the form:

$$l(\xi) = T\left[\ln\Gamma\left(\frac{v+1}{2}\right) - 0.5\ln\left[\pi(v-2)\right] - \ln\Gamma\left(\frac{v}{2}\right)\right] - 0.5\sum_{t=1}^{T}\left[(v+1)\ln\left(1+\frac{z^2}{v-2}\right) + \ln\left(h_t^2\right)\right]$$

where $\xi = (\mu, \gamma_0, \gamma_1, \gamma_2, \gamma_3, v)$ denotes the parameter vector of the EGARCH-Student-t (EGARCH-ST), $\Gamma(.)$ is the gamma function and v is the parameter that controls the distribution tails $(2 < v \le \infty)$.

Generalized Error Distribution

For the standardized generalized error distribution, the log-likelihood function takes the form:

$$l(\xi) = \sum_{t=1}^{T}\left[\ln\left(\frac{v}{\kappa_v}\right) - \left(1+v^{-1}\right)\ln 2 - 0.5\left|\frac{z_t}{\kappa_v}\right|^v - \ln\Gamma\left(v^{-1}\right) - 0.5\ln\left(h_t^2\right)\right]$$

where $\xi = (\mu, \gamma_0, \gamma_1, \gamma_2, \gamma_3, v)$ denotes the parameter vector of the EGARCH-Generalized error (EGARCH-GED), $\Gamma(.)$ is the gamma function, v is the tail-thickness parameter $(0 < v < \infty)$ and $\kappa_v = \sqrt{\left(\frac{2^{-2/v}\Gamma\left(v^{-1}\right)}{\Gamma\left(3v^{-1}\right)}\right)}$. The special cases of the generalized error distribution (GED) are the standard normal distribution when $v=2$, Laplace distribution when $v=1$ and uniform distribution when $v\to\infty$.

Skew Normal Distribution

For the standardized skew normal distribution, the log-likelihood function takes the form:

$$l(\xi)=T\ln(2\sigma)+\sum_{t=1}^{T}\left\{\ln\left[\phi(z_t\sigma+\mu)\right]+\ln\left(\Phi\left[(z_t\sigma+\mu)\alpha\right]\right)-0.5\ln\left(h_t^2\right)\right\}$$

where $\xi=(\mu,\gamma_0,\gamma_1,\gamma_2,\gamma_3,\alpha)$ denotes the parameter vector of the EGARCH-Skew Normal (EGARCH-SN), α is the skew parameter, μ and σ are the mean and standard deviation of the skew normal distribution, respectively.

Skew Student-t Distribution

For the standardized skew Student-t distribution, the log-likelihood function takes the form:

$$l(\xi)=T\left[\ln\Gamma\left(\frac{v+1}{2}\right)+\left(\frac{2}{\beta+1/\beta}\right)+\ln s-0.5\ln\left[\pi(v-2)\right]-\ln\Gamma\left(\frac{v}{2}\right)\right]$$
$$-0.5\sum_{t=1}^{T}\left\{(1+v)\ln\left[1+\frac{(sz+m)^2\beta^{-2I_t}}{v-2}\right]+\ln\left(h_t^2\right)\right\}$$

where $\xi=(\mu,\gamma_0,\gamma_1,\gamma_2,\gamma_3,v,\beta)$ denotes the parameter vector of the EGARCH-Skew Student-t (EGARCH-SST), v is the degree of freedom, $\Gamma(.)$ denote the gamma function, β is the skew parameter, and:

$$s=\sqrt{\left(\beta^2+\frac{1}{\beta^2}-1\right)-m^2},\quad m=\frac{\Gamma\left(\frac{v+1}{2}\right)\sqrt{v-2}}{\sqrt{\pi}\,\Gamma\left(\frac{v}{2}\right)}\left(\beta-\frac{1}{\beta}\right),\quad I_t=\begin{cases}1 & if \quad z_t\ge-\frac{m}{s}\\ -1 & if \quad z_t<-\frac{m}{s}\end{cases}.$$

Skew Generalized Error Distribution

For the standardized skew generalized error distribution the log-likelihood function takes the form:

$$l(\xi)=T\ln C+\sum_{t=1}^{T}\left(-\frac{1}{\left[1-sign(z-\kappa)\eta\right]^{v}\varphi^{v}}|z-\kappa|^{v}-0.5\ln\left(h_t^2\right)\right)$$

where $\xi=(\mu,\gamma_0,\gamma_1,\gamma_2,\gamma_3,v,\eta)$ denotes the parameter vector of the EGARCH-Skew Generalized error (EGARCH-SGED), v is the degrees of freedom, η is the skew parameter $(-1<\eta<1)$ and:

$$C = \nu\left[2\varphi\Gamma\left(\nu^{-1}\right)\right]^{-1}$$

$$\kappa = 2\eta AS\left(\eta\right)^{-1}$$

$$A = \Gamma\left(2/\nu\right)\Gamma\left(1/\nu\right)^{-\frac{1}{2}}\Gamma\left(3/\nu\right)^{-\frac{1}{2}}$$

$$\varphi = \Gamma\left(1/\nu\right)^{\frac{1}{2}}\Gamma\left(3/\nu\right)^{-\frac{1}{2}}S\left(\eta\right)^{-1}$$

$$S\left(\eta\right) = \sqrt{1+3\eta^2-4A^2\eta^2}$$

Generalized Hyperbolic Distribution

For the standardized generalized hyperbolic distribution, the log-likelihood function takes the form:

$$\begin{aligned} l\left(\xi\right) = {} & \lambda/2\ln\left(\alpha^2-\beta^2\right)+0.5\sum_{t=1}^{T}\left(\lambda-\frac{1}{2}\right)\ln\left(\delta^2+\left\langle z_t-\mu\right\rangle^2\right)-0.5\ln\left(2\pi\right)-\left(\lambda-\tfrac{1}{2}\right)\ln\alpha \\ & -\lambda\ln\delta-\ln\left[\Psi_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)\right]+\ln\left\{\Psi_{\lambda-\frac{1}{2}}\left[\alpha\sqrt{\delta^2+\left(z-\mu\right)^2}\right]\right\} \\ & +\left(\beta\left\{z-\mu\right\}\right)-0.5\sum_{t=1}^{T}\ln\left(h_t^2\right) \end{aligned}$$

where $\xi = \left(\mu,\gamma_0,\gamma_1,\gamma_2,\gamma_3,\beta,\lambda,\alpha\right)$ denotes the parameter vector of the EGARCH-Generalized Hyperbolic (EGARCH-GHB), δ is scale parameter, μ is location parameter, β is the asymmetry parameter, λ, α are real parameters, Ψ_λ is the modified Bessel function of third order.

Johnson Reparametrized (SU) Distribution

For the standardized Johnson reparametrized (SU) distribution, the log-likelihood function takes the form:

$$l\left(\xi\right) = T\ln\theta-T\ln\eta-0.5\sum_{t=1}^{T}\ln\left[1+\left(\frac{z_t-\tau}{\eta}\right)^2\right]+\sum_{t=1}^{T}\ln\left\{\phi\left[\upsilon+\theta\sinh^{-1}\left(\frac{z_t-\tau}{\eta}\right)\right]\right\}-0.5\sum_{t=1}^{T}\ln\left(h_t^2\right)$$

where $\xi = \left(\mu,\gamma_0,\gamma_1,\gamma_2,\gamma_3,\upsilon,\theta\right)$ denotes the parameter vector of the EGARCH-Johnson SU (EGARCH-JSU), ϕ is the density function of $N\left(0,1\right)$, τ, η are location and scale parameters, υ, θ denote the skew and kurtosis parameters.

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