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Applications of the linear Differential Equations on the plane and Elements of Nonlinear Systems, In Economics

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Abstract

In recent years, it has become increasingly important to incorporate explicit dynamics in economic analysis.

These two tools that mathematicians have developed, differential equations and optimal control theory, are probably the most basic for economists to analyze dynamic problems.

In this paper I will consider the linear differential equations on the plane (phase diagram) and elements of nonlinear systems, when we have unequal real roots of the same signs and opposite signs of characteristic roots, and the applications of the theory of differential equations to certain macroeconomic problems. The basic tools for discussion are phase diagram techniques.

Keywords: Differential Equations, Linear Systems, Nonlinear Systems, Phase Diagram (Plane), Liapunov's Theorem.

Introduction

Differential equations appear frequently in mathematical models that attempt to describe real-life situations. Many natural laws and hypotheses can be translated via mathematical language into equations involving derivatives. For example, derivatives appear in physics as velocities and accelerations, in geometry as slopes, in biology as rates of growth of populations, in psychology as rates of learning, in finance as rates of growth of investments and in economics as rates of change of the cost of living.

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The study of linear differential equations for the two-dimensional case is important, because many applications of differential equations are often two-dimensional and a number of key concepts such as "node", "saddle point", "spiral point", and "center" (which do not appear in one-dimensional problems) appear in two-dimensional problems. Also, the linear system for the two-dimensional case is often used as an approximation of the nonlinear system, and hence it is important to study the linear system, and understand the circumstances in which such an approximation is possible.

Nonlinear differential equations and systems of nonlinear differential equations occur frequently in applications. However, only a few types of nonlinear differential equations (for example, separable, homogenous, exact) can be solved explicitly. The same is true for nonlinear systems.

There are many applications of the theory of differential equation systems in economic problems. A simple illustration of the application to macroeconomics may be found in the stability of macro equilibrium. Nikaido (1972) provided a useful application of the nonlinear systems in the Tobin's seminal work (1965), incorporating money into the Solow-Swan growth model, the "money and growth" model. Takayama and Drabicki (1985) discussed in the neoclassical growth model with money and used a nonlinear differential system and phase diagram for providing a stable steady state. Furthermore, in dynamic programming and control problem, that are powerful tools in related problem analysis, linear and nonlinear differential systems can be applied.

This paper is organized as follows. In section one, linear system and dynamic behavior of the solution on the plane with unequal real roots of the same signs and opposite signs, has discussed. Section two is about nonlinear system, local behavior of the trajectories on the plane and stability of the nonlinear system Liapunov's direct model.

And in the last section, application of the theory of differential equation systems in IS-LM model is illustrated by using phase diagram techniques.

1- The Phase Plane: Linear Systems

Since many differential equations cannot be solved conveniently by analytical methods, it is important to consider what qualitative information can be obtained about their solutions without actually solving the equations.

In this section, I investigate the dynamic behavior of the solutions systems of linear differential equations on the plane.

I start with a secend order linear homogeneous system with constant coefficients. Such a system has the form (Boyce, 2000):

$$\frac{dX}{dt} = AX \tag{1}$$

Where A is a 2×2 constant matrix and X is a 2×1 vector. The solutions of Eq.(1) of the form are:

$$X = \xi e^{rt} \tag{2}$$

Upon canceling the nonzero scalar factor e^{rt} it can be obtained

$$(A - rI)\,\xi = 0 \tag{3}$$

To solve the system of differential equation (1), we must solve the system of algebraic equations (3). This latter problem is precisely the one that determines the eigenvalues and eigenvectors of the matrix A. For the system one points where AX=0 correspond to equilibrium (constant) solutions and they are called critical points. Assuming that A is nonsingular, or that det A≠0. It follows that X=0 is the only critical point of the system (1).

In analyzing the system (1) several different cases must be considered, depending on the nature of the eigenvalues of A, that I consider two major cases in this section.

1-1- Real Unequal Eigenvalues of the Same Sign

The general solution of Eq.(1) is

$$X = C_1 \xi^{(1)} e^{r_1 t} + C_2 \xi^{(2)} e^{r_2 t} \tag{4}$$

Where r_1 and r_2 are either both positive or negative. Suppose first that $r_1 < r_2 < 0$, and that the eigenvectors $\xi^{(1)}$ and $\xi^{(2)}$ are as shown in Figure 1.a. It follows from Eq.(4) that $X \to 0$ as $t \to \infty$ regardless of the values of C_1 and C_2; in other words, all solutions approach the critical point at the origin as $t \to \infty$. If the solution starts at an initial point on the line through $\xi^{(1)}$, then C_2=0.

Consequently, the solution remains on the line through $\xi^{(1)}$ for all t, and approaches the origin as $t \to \infty$. Similarly, if the initial point is on the line through $\xi^{(2)}$, then the solution approaches the origin along that line. In the general situation, it is helpful to rewrite Eq.(4) in the form:

$$X = e^{r_2 t}\left[C_1 \xi^{(1)} e^{(r_1 - r_2)t} + C_2 \xi^{(2)}\right] \qquad (5)$$

Observe that $r_1 - r_2 < 0$. Therefore, as long as $C_2 \neq 0$, the term $C_1 \xi^{(1)} \exp[(r_1 - r_2)t]$ is negligible compared to $C_2 \xi^{(2)}$ for t sufficiently large. Thus, as $t \to \infty$, the trajectory not only approaches the origin, it also tends toward the line through $\xi^{(2)}$. Hence all solutions approach the critical point tangent to $\xi^{(2)}$ except for those solutions that start exactly on the line through $\xi^{(1)}$. Several trajectories are sketched in figure 1.a. Some typical graphs of x_1 versus t are shown in figure 1.b, illustrating that all solutions exhibit exponential decay in time. The behavior of x_2 versus t is similar. This type of critical point is called a node or a nodal sink.

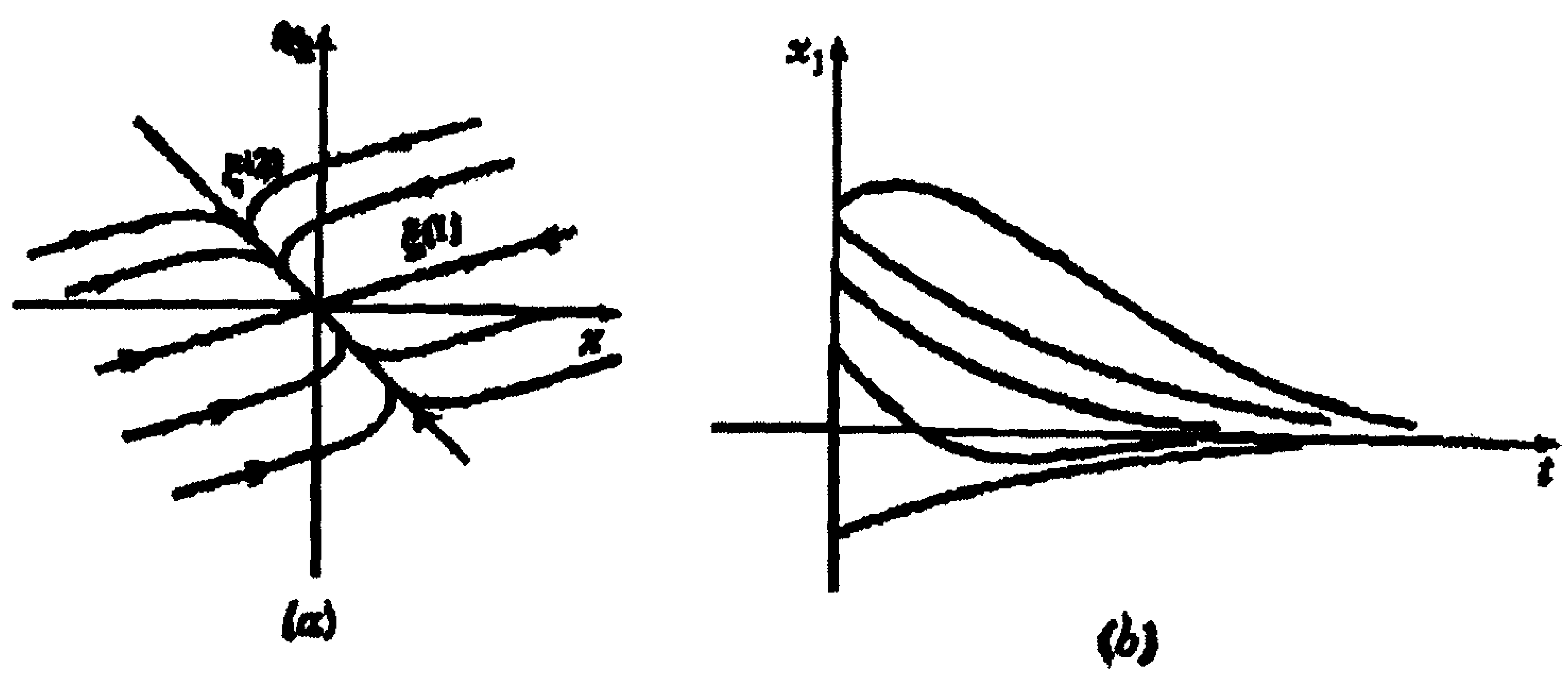

Figure 1: A node; $r_1 < r_2 < 0$ (a) The Phase Plane. (b) x_1 versus t.

1-2- Equal Eigenvalues

Now suppose that $r_1 = r_2 = r$. Considering the case in which the eigenvalues are negative. There are two subcases, depending on whether the repeated eigenvalue has two independent eigenvectors or only one.

(a) Two independent eigenvectors

The general solution of Eq.(1) is

$$X = C_1 \xi^{(1)} e^{rt} + C_2 \xi^{(2)} e^{rt} \qquad (6)$$

Where $\xi^{(1)}$ and $\xi^{(2)}$ are the independent eigenvectors. The ratio $\frac{x_2}{x_1}$ is independent of t, but depends on the components of $\xi^{(1)}$ and $\xi^{(2)}$, and on the arbitrary constants C_1 and C_2. Thus every trajectory lies on a straight line through the origin, as shown in Figure 2.a. Typical graphs of x_1 or x_2 versus t are shown in Figure 2.b. The critical point is called a proper node, or sometimes a star point.

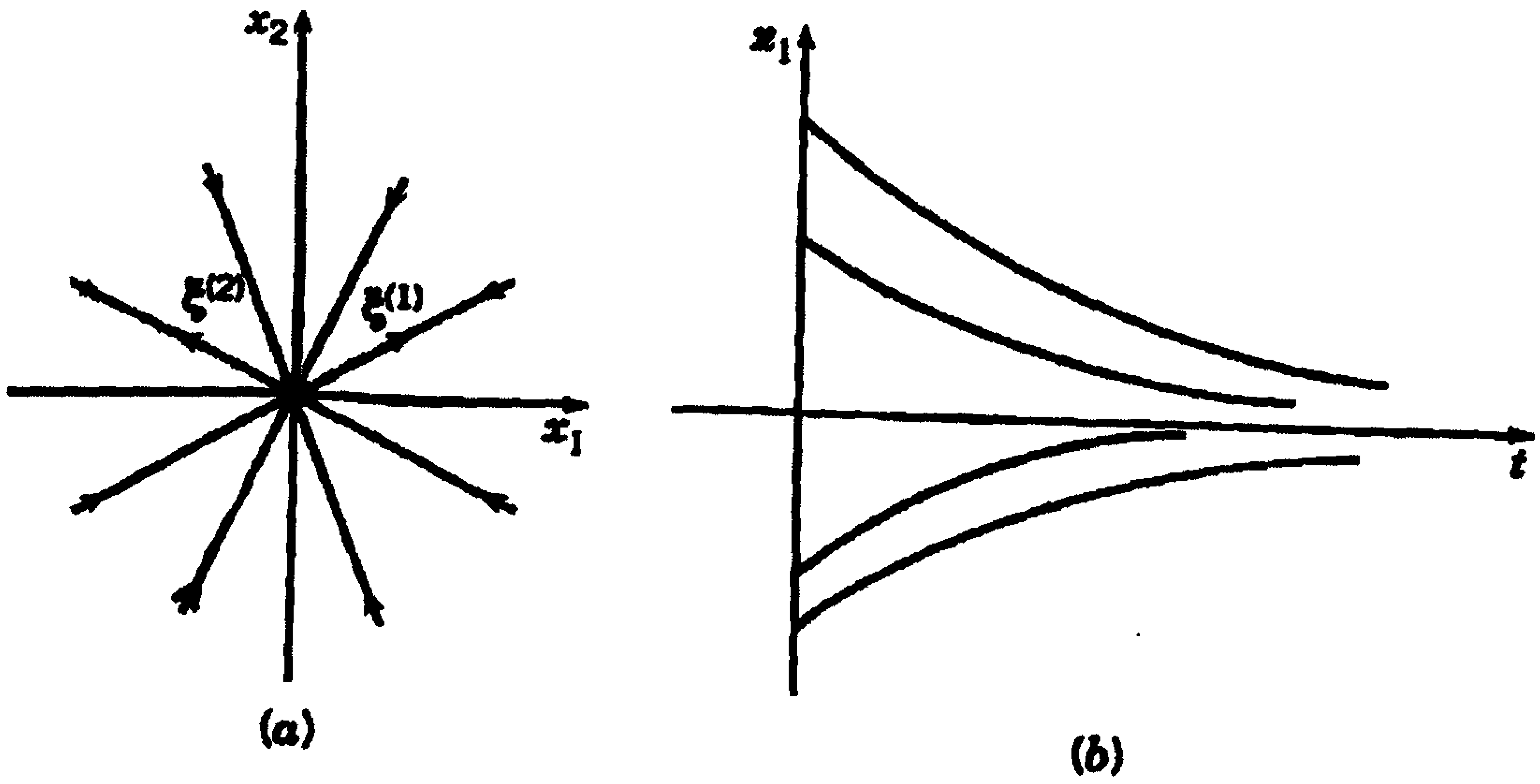

Figure 2: A proper node, two independent eigenvector $r=r_1=r_2<0$
(a) The phase plane. (b) x_1 versus t.

(b) One independent eigenvector

The general solution of Eq.(1) in this case is:

$$X=C_1\xi e^{rt}+C_2(\xi t e^{rt}+\eta e^{rt}) \quad (7)$$

Where ξ is the eigenvector and η is generalized eigenvector associated with the repeated eigenvalue.

The orientation of the trajectories depend on the relative positions of ξ and η. One possible situation is shown in Figure 3. a. To locate the trajectories it is helpful to write the solution (7) in the form.

$$X=[(C_1\xi+C_2\eta)+C_2\xi t]e^{rt}=ye^{rt} \quad (8)$$

To sketch the trajectory corresponding to a given pair of values of C1 and C2 it can be proceed in the following way. First, draw the line given by $(C_1\xi + C_2\eta) + C_2\xi t$ and note the direction of increasing t on this line. Two such lines are shown in Figure 3.a, one for $C_2>0$ and the other for $C_2<0$. Next note that the given trajectory passes through the point $C_1\xi + C_2\eta$ when t=0. Further, as t increases, the direction of the vector X given by Eq.(8) follows the direction of increasing t on the line, but the magnitude of X rapidly decreases and approaches zero because of the decaying expotential factor e^{rt}. Finally, as t decreases toward -∞ the direction of X is determined by points on the

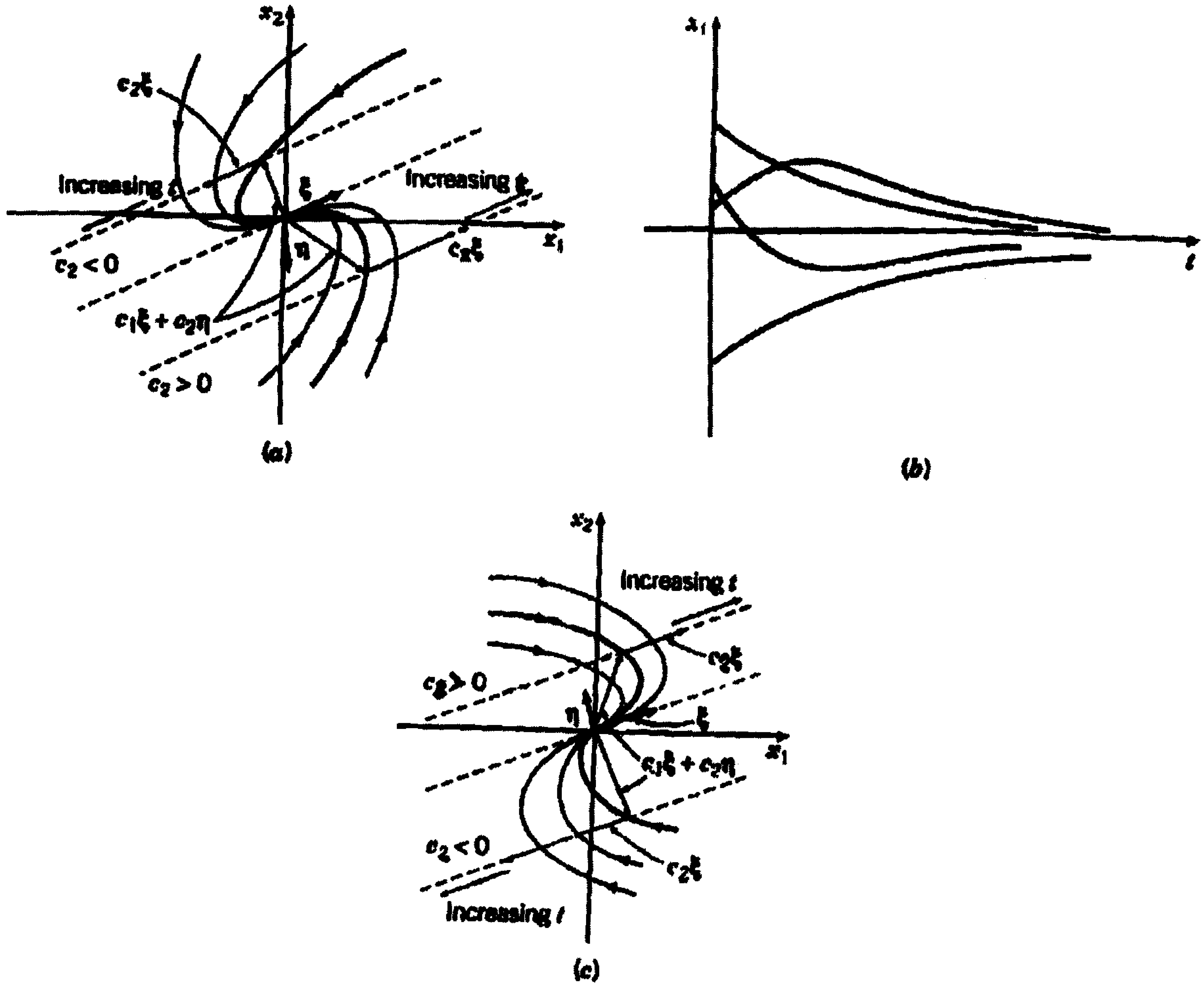

Figure 3: An improper node, one independent eigenvector; $r_1 = r_2 < 0$ (a) the phase plane. (b) x1 versus t. (c) The phase plane

corresponding part of the line and the magnitude of X approaches infinity, thus obtaining the heavy trajectories in Figure 3.a. A few other trajectories are lightly sketched as well to help complete the diagram. Typical graphs of X_1 versus t are shown in Figure 3.b.

The other possible situation is shown in Fig 3.c, where the relative orientation of ξ and η is reserved. As indicated in this Figure, this results in a reversal in the orientation of the trajectories.

2- Nonlinear Systems

2-1- Local Behavior of the Trajectories on the Plane

Consider a nonlinear system, (Takayama 1994)

$$\dot{x}=f_1(x,y) \quad , \quad \dot{y}=f_2(x,y), \tag{9}$$

Where f_1 and f_2 are assumed to be continuously differentiable.
Furthermore, assume that the origin is an equilibrium point of (1), i.e,

$$f_1(0,0)=0 \text{ and } f_2(0,0)=0$$

Notice that the equilibrium point in the system (9) may not be unique, therefore assume that there exists a neighborhood (or a circle) about the origin in which there are no other equilibrium points.

Now expand $f_1(x,y)$ and $f_2(x,y)$, in a Taylor series about the origin. Then

$$\dot{x}=a_{11}x+a_{12}y+\dot{f}_1(x,y) \quad , \quad \dot{y}=a_{21}x+a_{22}y+\dot{f}_2(x,y) \tag{10}$$

Where $a_{i1}\equiv\frac{\partial f_i(0,0)}{\partial x}$ and $a_{i2}\equiv\frac{\partial f_i(0,0)}{\partial y}$, i=1,2 , and the functions $\dot{f}_1(x,y)$ and $\dot{f}_2(x,y)$, signify the second-order or higher order terms. From the differentiability of the functions f_1 and f_2, the result is,

$$\dot{f}_1(x,y)=0(z) \ , \ \dot{f}_2(x,y)=0(z), \text{ where } z\equiv(x,y) \tag{11}$$

Here 0(z) is "Landau's O" that is,

$$\frac{0(z)}{\|z\|}\to 0 \quad as \quad \|z\|\to 0.$$

Note that this implies $\dot{f}_1(0,0)=0$ and $\dot{f}_2(0,0)=0$, so that (0,0) is indeed an equilibrium point of (10). Assume that

$$a_{11}a_{22}-a_{21}a_{12} \neq 0 \quad (12)$$

The nonlinear system (9) that satisfies conditions (11) and (12) is sometimes called an "almost linear system" in neighborhood of the equilibrium point (0,0).

It would now be natural to conjecture that the qualitative behavior of the paths of (9) near the equilibrium point is similar to that of the paths of the related linear systems:

$$\dot{x}=a_{11}x+a_{12}y \quad , \quad \dot{y}=a_{21}x+a_{22}y \quad (13)$$

Where the a_{ij}'s are taken from (10). The procedure for obtaining the linear system (13) from the nonlinear system (9) is called "linearization".

2-2- Stability of the Nonlinear System Liapunov's Direct Method

In this section I discuss about stability for the nonlinear system for the n-dimensional case. The method is known as "Liapunov's second method" or "direct method", which is based on liapunov (1907) (Takayama 1995). Liapunov's (direct) method provides a more global type of information, such as the asymptotic stability of an equilibrium point. It is facilitated by constructing an auxiliary function, and it does rely on the linearization of the original nonlinear system.

Since Liapunov's method is not confined to the two-dimensional case, the system of n first-order nonlinear differential equations can be considered as follows:

$$\dot{x}=f(x) \quad (14)$$

The system (14) is a compact form of

$$\dot{x}_i=f(x_1,x_2,\ldots,x_n) \quad , \quad i=1,2,\ldots,n$$

Let the origin be an equilibrium point of (13), so
$f(0)=0$.

The heart of Liapunov's direct method is to construct a real-valued function V(x) in which x is governed by (13). Now state the major theorem,

Theorem 3 (Liapunov's Direct Method)

Suppose that there exists a real-valued, continuously differentiable function V(x) for the system (13) such that:

(i) $V(x) > o$ for all $x \neq 0$, and $V(0)=0$,

(ii) $dV\left[\frac{x(t)}{dt}\right] < 0$ for all x that satisfy (13), and

(iii) $V(x) \rightarrow \infty$ as $\| x \| \rightarrow \infty$

Then $x^* = 0$ is asymptotically globally stable, that is, $x(t) \rightarrow \infty$ as $t \rightarrow \infty$, regardless of the initial condition.

The function V(x) is called the Liapunov function. The real valued, continues function that has the property of condition (i) of theorem 3 is called "Positive definite".

3- Applications

As already stated, there are many applications of the theory of differential equation systems in economic problems.

For example Eisner and Strots (1963) were considered net investment as a process that expands a firm's plan size and solved the model by phase diagram.

Hartl R. (1983) has derived qualitative properties of the optimal policy for a class of nonlinear optimal control problems. The problem structure was characteristic of certain economic investment problems where two control instruments influenced the rate of deterioration of a capital good. A stability analysis in both of the planes state- costate and state-control was carried out.

Eicher.S. and Turnovsky J. (2001) have shown how, under plausible conditions, the transitional dynamics in a two- sector R&D-based non- scale growth model are represented by a two-dimensional stable saddle path.

Dieci. R., Bischi, G.I. and Gardinin. L. (2001) were considered a discrete-time economic model where the savings are proportional to income and investment demand depends on the difference between the current income and its exogenously assumed equilibrium level, through a nonlinear S-Shaped increasing function and they also analyzed how changes in the parameters' values modify both the asymptotic dynamics of the system and the structure of the basins of the different and often coexisting attractors in the phase-plane.

In this section one of the applications of the theory of differential equations to certain macroeconomic problems is illustrated. The basic tools are phase diagram techniques, the theory of linear differential equations on the plane.

3-1- Static Macro Equilibrium and Its Stability

A simple illustration of the applications to macro economics may be found in the stability of macro equilibrium and its stability. The IS-LM macro adjustment process can be described by (Takayama 1994):

$$\dot{y}=c[E(y-T,r)+G-y]\equiv Q_1(y,r), \tag{15}$$

$$\dot{r}=d\left[L(y-r)-\frac{M}{p}\right]\equiv Q_2(y,r), \tag{16}$$

Where y=output, r=interest rate, E=consumption plus investment expenditures, G= government expenditures, T=taxes minus transfer payments, L=money demand, M=money supply, and P=price level.

The parameters c and d, respectively, signify the speeds of adjustment for the goods market and money market. The macro equilibrium may be defined by (y^*, r^*) that satisfies

$$Q_1(y^*, r^*) = 0 \quad \text{and} \quad Q_2(y^*, r^*) = 0 \tag{17}$$

Which is nothing but the point of intersection of the IS and the LM curves.

To investigate the trajectory of (15), (16), the following assumptions are imposed:

$$0<E_y<1,\ E_r<0, L_y>0, L_r<0, \text{ for all y and r}, \tag{18}$$

Where $Ey\equiv\frac{\partial E}{\partial(y-T)}$, $E_r\equiv\frac{\partial E}{\partial r}$, etc. Therefore:

$$Q_{1y} = C(E_y - 1) < 0 \quad , \quad Q_{1r} = CE_r < 0, \tag{19}$$

$$Q_{2y} = dL_y > 0 \quad , \quad Q_{2r} = dL_r < 0, \tag{20}$$

For all y and r , where $Q_{1y} \equiv \frac{\partial Q_1}{\partial y}$, $Q_{1r} \equiv \frac{\partial Q_1}{\partial r}$, etc. If investment and consumption are completely inelastic, $E_r = 0$ that $Q_{1r} = 0$. Also, under a liquidity trap $L_r \to -\infty$ so that $Q_{2r} \to -\infty$. Hence, for such polar cases there are:

$$Q_{1r} = 0 \text{ and / or } Q_{2r} \to -\infty, \text{ as well as } Q_{1y} < 0 \ , \ Q_{2y} > 0 \tag{21}$$

The slopes of the $(Q_1 = 0)$ and $(Q_2 = 0)$-curve s may, respectively, be obtained as:

$$\frac{dr}{dt}\Big|_{Q_1=0} = \frac{-Q_{1y}}{Q_{1r}} < 0 \ , \ \frac{dr}{dy}\Big|_{Q_2=0} = \frac{-Q_{2y}}{Q_{2r}} > 0 \tag{22}$$

Thus under assumption (18), the $(Q_1 = 0)$ – curve is downward sloping, and the $(Q_2 = 0)$ – curve is upward sloping. This is illustrated in figure 3. These two curves, respectively, correspond to familiar IS and LM curves. Also from (19) and (20), we may conclude that y*>0 to the left of the $(Q_1 = 0)$ – curve, and r*>0 to the right of the $(Q_2 = 0)$ – curve.

Consider the linear approximation system of (15) and (16):

$$\dot{y} = Q_{1y}^{*}(y - y^{*}) + Q_{1r}^{*}(r - r^{*}), \tag{23}$$

$$\dot{r} = Q_{2y}^{*}(y - y^{*}) + Q_{2r}^{*}(r - r^{*}), \tag{24}$$

Where the asterisk (*) signifies that the partial derivatives are evaluated at (y*,r*).

As discussed in the previous section, the shape of trajectories crucially depends upon the eigenvalues of the corresponding linear approximation system. The characteristic equations of (23) and (24) can be written as:

$$\lambda^2 - \alpha\lambda + \beta = 0 \tag{25}$$

Where:

$$\alpha = Q_{1y}^* + Q_{2r}^* \quad \text{and} \quad \beta = Q_{1y}^* Q_{2r}^* - Q_{2y}^* Q_{1r}^* \tag{26}$$

and the eigenvalues are obtained as:

$$\lambda_1, \lambda_2 = \frac{1}{2}\left[\alpha \pm \sqrt{\alpha^2 - 4\beta}\right] \tag{27}$$

Where:

$$\begin{aligned} \alpha^2 - 4\beta &= (Q_{1y}^* + Q_{2r}^*)^2 - 4(Q_{1y}^* Q_{2r}^* - Q_{2r}^* Q_{1y}^*) \\ &= \left[C(1 - E_y^*) - dL_r^*\right]^2 + 4cdL_y^* E_r^* \end{aligned} \tag{28}$$

Since $\alpha<0$, λ_1 and λ_2 cannot be pure imaginary. Hence assuming away the Knife- edge case of $\alpha^2 - 4\beta = 0$, the behavior of the trajectory of (15) and (16) in a neighborhood of (y* , r*) can be approximated by that of its linear approximation system (15) , (16). It is a spiral point if and only if:

$$\left[C(1 - E_y^*) - dL_r^*\right]^2 + 4cdL_y^* E_r^* < 0 \tag{29}$$

and it is a node if

$$\left[C(1 - E_y^*) - dL_r^*\right]^2 + 4cdE_r^* L_y^* > 0 \tag{30}$$

Assuming that (29) and (30) hold, the trajectory (y,r) for (15), (16) in figure 4 can be illustrated, where (y*,r*) is a spiral point.

The "Keynesians", tend to maintain a low magnitude of E_r (where E_r=0 signifies completely interest-inelastic consumption and investment) and / or a high magnitude of L_r (where $L_r \to -\infty$ signifies the liquidity trap). Under such circumstances, inequality of (30) is likely to hold, and the equilibrium point would

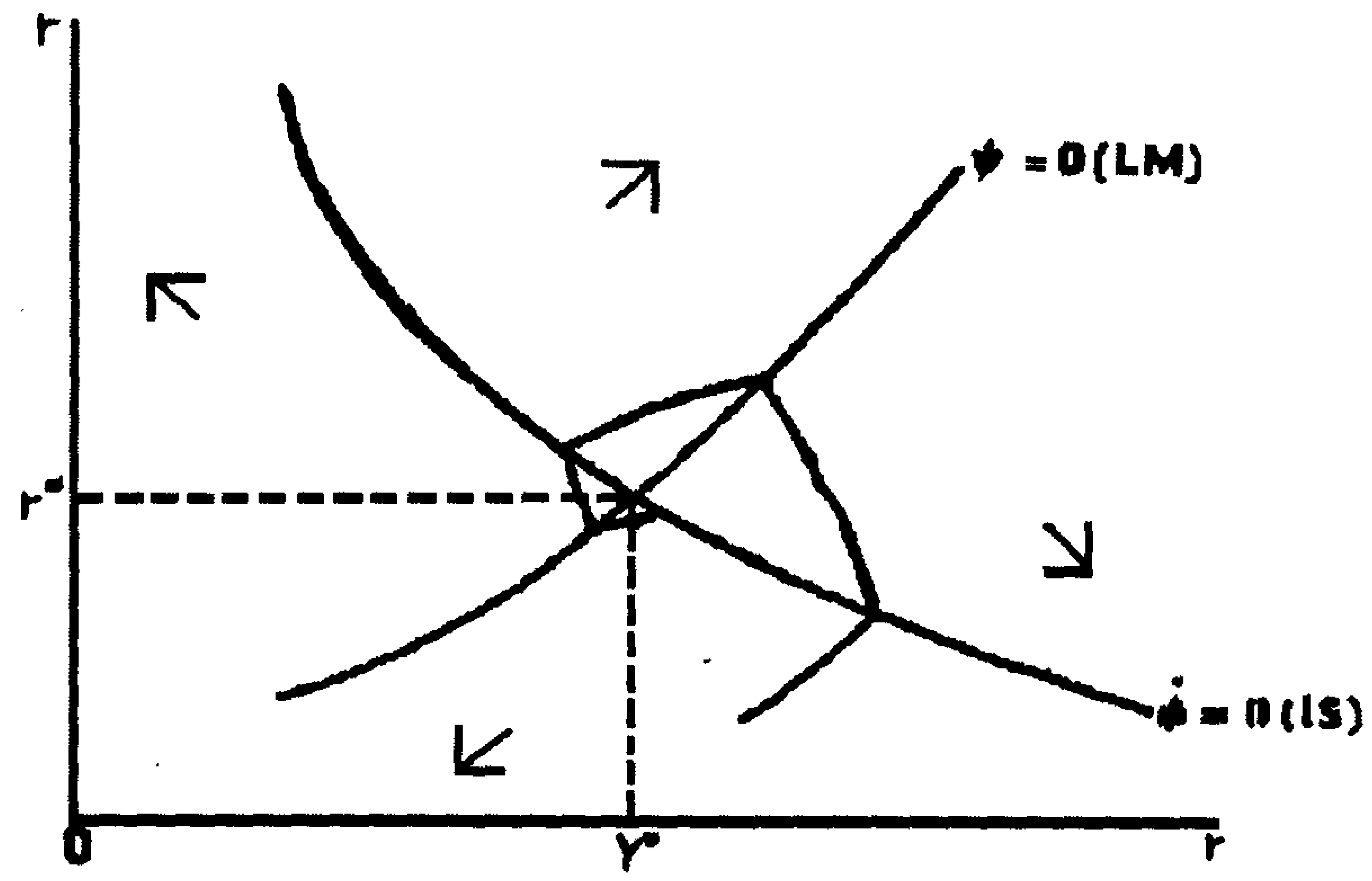

Figure 4: The adjustment path of statues macro equilibrium : the case of a spiral point

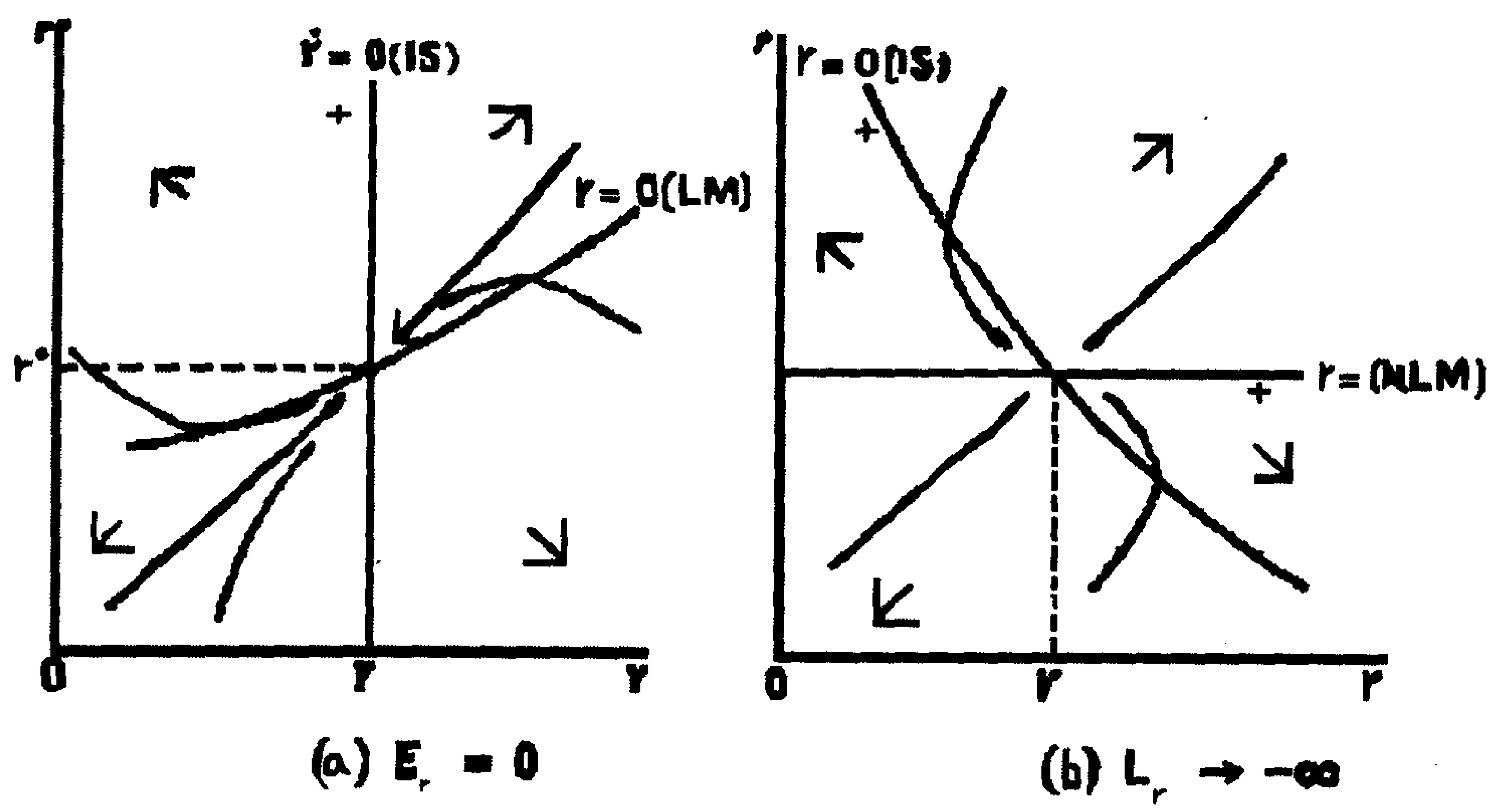

Figure 5: Keynesian adjustment paths: extreme cases

be a node. When E_r=0 the IS curve is vertical, while when $L_r \rightarrow -\infty$, the LM curve is horizontal.

The adjustment paths of these two cases are illustrated in figure 4, where the equilibrium point is asymptotically globally stable.

These results can be summarized as follows:

Proposition

Under the monetarist inclination, the macro equilibrium point (y*,r*) would be a "spiral point", while under the Keynesian inclination it would be a "node". The trajectories of these cases are illustrated in figure 5.

4- Conclusion

In recent years, it has become increasingly important to incorporate explicit dynamic in economic analysis.

The two tools that mathematicians have developed, differential equations and optimal control theory, are probably the most basic for economists analyzing dynamic problems.

In many of the economics problems, the theory of differential equations can be applied. In this article at first, I review the linear differential equation on the plane (phase diagram) and nonlinear systems, when we have unequal real roots of the same signs and opposite signs of characteristic roots. Then I considered the application of the theory of differential equations to certain macroeconomic problems via phase diagram techniques. Then, the IS-LM macro adjustment was considered by the plane and phase diagram, and its result is that, under the monetarist inclination, the macro equilibrium point would be a spiral point, while under the Keynesian inclination it would be a node.

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