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Underground Economy and Tax Gap

Ali Taiebnia*
Shapour Mohammadi**

Abstract

In this paper the relation of underground economy and tax gap in Iran has been examined. The results show that the underground economy size is almost 20 percent of GDP in Iran. In this research three methods of estimation of underground economy size has been used and combined to achieve a more precise estimation. The methods that have been used are: cash, fuzzy and latent variable methods. The size of tax gap in Iran has been over 30 billion dollars in recent years. Some new payment systems as ATM cause change in money velocity which makes it necessary that new factors to be taken into account

Keywords: Underground Economy Size, Fuzzy Sets, Forecast Combination Informal Economy Size, Latent Variable method.

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*Assistant professor at University of Tehran, Faculty Economic.
** Assistant professor at University of Tehran, Faculty of Management.

1- Introduction

Underground economy is one of the unwanted facts in many countries. This sector can be defined as legal and illegal sector of economies. Households which produce goods and services for their own uses are agents of informal economy but not illegal agents. Black market economy, shadow economy, and underground economy are almost equal in terms of literature. However, they are different from informal economy and illegal ones. The first term has a broader definition than the second one. Economic literature is familiar with the story of underground or shadow economy. Tanzi (1980) is the earliest work on the estimation underground economy size. His work is based on liquidity demand. another study in this area is done by Frey and Hanlor(1984) that uses unobservable variables for estimation of underground economy (UE hereafter) size. They find variables such as tax burden, regulation and tax morality as statistically significant variables. This study applies Multiple Indicators and Multiple Causes (MIMIC) which is a factor analysis method for estimating UE as an unobservable variable. From an econometrics point of view linear structural equations (LISREL) is a generalization of MIMIC model. LISREL has two parts: measurement models and structural equations. Measurement model relates unobserved variables to observable indices and structural equations focus on the relation between unobservable variables. In the Frey and Hanlor, the unobserved variable is UE which its size is determined by exogenous variables which are measured without any error. Another set of variables are indicators that affect UE size. Other studies in this area, Giles and Caragata (1994), and Caragata and Giles (1994) examine the response of underground economy to tax burden and tax mix. In addition, Schneider and Enst (2000) estimated UE size in developed and developing countries by various methods.

The rest of the paper is organized as follows. The second section briefly explains methods of estimation and provides our main method in the estimation of UE size. Third section presents estimation and interpretation of UE size as well as tax gap in Iran. The last section is devoted to concluding remarks and policy prescriptions.

2- Methods of Estimation for Underground Economy Size

Various methods can be used for the estimation of UE size. The methods are mainly based on liquidity demand methods mainly and in some cases energy demand and national accounts imbalances. For example, demand for money is a method, which supposes that transactions in underground economy take place with money. According to this method other medium of exchange can not be used in illegal or informal economy. Other methods, such as MIMIC model and fuzzy approach also need a quantitative estimate at first or at least in one of years for UE size. In this section, we explain these methods in detail.

Methods of estimation can be divided to two categories: direct and indirect methods. Direct methods are based on survey, tax auditing and voluntary questioner approach. These methods are micro approach and are implemented in various countries. The main advantage of these methods is ability of them in providing broad range of information about structure of underground economy. However, the results depend on formulation of questioner and willingness of people to give exact and correct answers. One of helpful methods in estimating UE size is difference between actual and declared (stated) tax. The main weakness of this method is its sample selection bias. When tax auditing's focus is on tax payers, tax avoiders are ignored and therefore the estimation of underground economy is only a part of real underground economy and can be used as minimum size of UE in practice.

Indirect approaches are macro approaches mainly and usually are called index approaches. These methods use macroeconomic and other factors for estimation of underground economy size. One of the methods is defined as difference between national income and national expenditure. These methods can be reliable if these two methods are used independently and measurement error is not high. However, in practice both problems are present. In most of countries difference between income and expenditure is used for adjustments of national accounts which makes less difference between these figures in practice and therefore are not reliable information for UE size measurement. Also some sectors are omitted from national income accounting and other sectors are measured with error, therefore statistics of national accounts can nit be used in estimation of UE size

(Schneider and Enste, 2000, p.93). For example, UE size of Russia was reported 74.9 percent for 1995.

The second method is difference between formal and informal labor force. Change of labor in the formal sector with given rate of participation can be attributed to change of UE size. This method has at least two drawbacks. The first one is other factor than change in UE size which may change labor of formal sector. The second one is possibility of working in both formal and informal sector simultaneously.

The Third method is transaction method which uses Fisher equation $MV = PT$, where M is money, V is velocity of money circulation, P stands for prices and T is total transaction. The method is proposed by Feige (1979). In this method supposed nominal GNP is composed from formal and informal GNP therefore subtracting formal GNP form the GNP that is calculated by Fisher equation one can obtain informal GNP. In this method argued that although the informal activities are not included in formal GNP, transactions show both formal and informal activities. In other words if formal sector's size is constant, changes in transactions are result of changes in informal activities. This method supposes that the size of underground economy is zero in the base year and the ratio of UE to formal economy is constant. These two assumptions and the necessity of having a lot of information cause restrictions, which make this method inapplicable.

The Fourth method is cash demand method. For the first time this method is used by Tanzi. The main reason behind this method is that informal activities mainly are done by cash, not by banking instruments such as check and drafts, etc. To extracting the effect of other variables such as interest rate, payment habits, income and so on, an econometric model should be estimated. Also some factors such as tax burden and regulations complexity which encourage economic agents to do their transaction in underground economy should be entered. The regression was used by Tanzi (Tanzi 1983) is as follows:

$$Ln(C/M_2)_t = \beta_0 + \beta_1 Ln(1+TW_t) + \beta_2 Ln(WS/Y)_t + \beta_3 Ln(R_t) + \beta_4 Ln(Y/N) + \varepsilon_t \quad (1)$$

With expected signs $\beta_1 > 0, \beta_2 > 0, \beta_3 < 0, \beta_4 > 0$

Where C/M2 is cash ratio to demand deposits and time deposits, TW weighted average of tax rate (a proxy for change in volume of underground economy), WS/Y share of salary and wage in national income (for taking to account various patterns of payment and cash holding), R is interest rate for cost of holding money and Y/N is per capita income. Excess demand for cash can be result of tax burden and other factors which make people to do transactions in underground economy. In this method the unrealistic assumption of zero value for underground economy is not necessary, however to use this method money velocity should be estimated for formal sector and be used for informal sector. This assumption that velocity of money in formal and informal sectors are the same is not so realistic. However, we cannot estimate velocity of money in the informal sector and there is no option other than accepting the same velocity for money in formal and informal sectors:

$$Ln(C\hat{/}M_2)_t = \hat{\beta}_0 + \hat{\beta}_1 Ln(1+TW_t) + \hat{\beta}_2 Ln(WS/Y)_t + \hat{\beta}_3 Ln(R_t) + \hat{\beta}_4 Ln(Y/N) + \varepsilon_t$$

With assumption of zero rate tax:

$$Ln(C\hat{/}M_2)_t^* = \hat{\beta}_0 + \hat{\beta}_2 Ln(WS/Y)_t + \hat{\beta}_3 Ln(R_t) + \hat{\beta}_4 Ln(Y/N) + \varepsilon_t \qquad (2)$$

Let α be the ratio of cash to deposits in the formal sector, then $C_t^* = M_{2t}e^{\alpha}$ and with subtracting C_t^* from C_t, one can get the size of underground economy $C_t^I = C_t - C_t^*$. In this stage we should estimate velocity of money (M2 here) and by the formula PT=MV size of underground economy can be estimated. This method is one of the common methods in the estimation of underground economy size, however some critiques of it are as follows:

1- Most of underground economy activities are done by cash, but not all of them. For example questioner researches show that only 80 percent of underground economy activities are done by cash in Norway. (Isachsen & Strom, 1980, 1985)

2- In this method only one of the factors that may lead to growth of shadow economy, tax burden, is considered, whereas there are some other important factors such as tax morality, and belief of tax payers about tax and government. Therefore, size of economy in this method is underestimated.

3- Increase in demand for cash is partially a result of decrease in demand deposits in some countries and cannot entirely be related to the underground economy.

4- The last critique is related to the assumption of constancy of money velocity.

The Fifth method is electricity consumption method which was proposed by Kaufmane and Kaliberda (1996), and Lacko (1998). In the method of Kaufmane and Kaliberda difference of electricity growth and GDP growth is used for extension of underground economy. Lacko method is based on household electricity consumption is used in shadow economy, therefore household consumption of electricity should be considered. There are many drawbacks in the use of electricity method, which make it useless for our study (Schneider, 2000), therefore avoid of further explanations.

Sixth method is related to underground variables, so called model approach. This method is also called Multiple Indicators and Multiple Causes (MIMC) approach. In this method unobserved variables such as UE size is related to observable variables such tax burden, regulation intensity and unemployment in formal sector. These variables are called causes. On the other side indicators are effects of unobservable variables. Effects can be GDP growth of formal sector, participation rate of formal sector and money variables. The demand for M1 is increases as UE size increases. This method is used by Frey and Hanneman (in 1984). (Further details of this methods is presented in Appendix A5)

We will use this method for Iran data in LISREL8 for calculation of UE size. Initial level of these estimates will be determined by liquidity approach.

Another method is called econometric method in the context of underground economy studies. This method is based on Baumol and Tobin "Optimal Cash Balance (1989)". The model is as follows:

$$M_t = \beta_0 Y_{Rt}^{\beta_1} R_t^{\beta_2} P_t^{\beta_3} \exp(\varepsilon_t) \tag{3}$$

Where M_t is money volume, Y_{Rt} is recorded real income or production, R_t is short term interest rate and P_t is general level of prices. This method is alternative form of money demand. Taking log and denoting logarithm of variables by lowercase letters:

$$m_t = \beta_0 + \beta_1 y_{Rt} + \beta_2 r_t + \beta_3 p_t + \varepsilon_t \qquad (4)$$

Where $\beta_0 = Ln\beta_0'$. If the underground sector is included in the model $Y_{RH_t} = (Y_{Rt} + Y_{Ht})$ the model can be stated as:

$$m_t = \beta_0 + \beta_1 y_{RHt} + \beta_2 r_t + \beta_3 p_t + \varepsilon_t \qquad (5)$$

In this method, using Y_{Ht} data which is calculated from other methods model is estimated and velocity of money is calculated for formal and underground sectors. This model is appropriate for the cases that one tends to represent the relation between macroeconomic variables when underground sector is considered as an economy sector. In other words, this method takes to account effects of putting aside underground sector in policy analysis.

Giles and Johnson (1999) extend Trandel-Snow model and examine relation between effective tax rate and the ratio of underground sector to GDP. They show that in nonconstant range of income and decreasing absolute risk aversion and nondecreasing relative risk aversion, an increases in average rate of can increase or decrease both of formal and informal sectors' size. Giles and Tedds (2000) study indicates that there is a positive S shape relation between effective tax rate and size of underground economy.

Seventh method which is used for modeling UE size is fuzzy logic approach. This method is introduced by professor Lotfi Zade. Fuzzy logic makes set theory more flexible and effective in real world applications. In classic set theory each element such as, a , may or may not be an element of set A . While in fuzzy sets each element of a set belongs to the set with a specific degree of membership. For example degree of membership to A for element , a , is 0.1. In other words in fuzzy sets each element is related to a degree of membership.

$$X = \{a, b, c, d\}$$

$$\widehat{X} = \{(a, 1), (b, 0.5), (c, 0.3), (d, 0.7)\}$$

Where X is a classic set and $\widehat{X}$ is a fuzzy set. In the fuzzy set each element such as a, b, c, d have a membership degree that states the degree of membership of that element in the set $\widehat{X}$. This approach in sets and logic makes approximate reasoning possible for effective modeling real world phenomena. Although, interval mathematics has most of strengths of fuzzy logic, fuzzy logic is very simple in practical applications. As father fuzzy sayes there are other methods that can be used instead of fuzzy logic, however it is one of simplest methods for similar problems.

The elements of fuzzy sets are similar to a classic set. For the degree of membership one can use common membership function in fuzzy logic literature. Appropriate membership function differs for different applications. In spite of probability, sum of membership degrees is not necessarily one and can be defined in an interval other than $[0,1]$.

The concepts such as less, medium, more, and also linguistic variables can be handled with fuzzy logic. Most of operators of set theory are applicable in fuzzy sets with a little adjustment. For example, union of sets can be stated with $\max$ of membership degrees and intersection of sets with $\min$ of membership degrees and complement with subtracting membership degree from 1. Therefore, Identity property, Commutative, Distribution, and DeMorgan rules are satisfied in fuzzy sets.

The fuzzy sets are applied to various problems which are not limited to specific area of science. One of the applications of fuzzy logic is measurement of unobservable variables. Giles is from among the first of researchers that applied fuzzy logic in study of size of underground economy. Further details are presented in Gills (1999).

When one forecasts economic variables with different methods, he can combine them to make a more accurate forecast. Methods like regression and variance-covariance of forecasts are the most common ones. In the first method, actual value of forecasted variable is given for some periods and one should determine which weights of various forecasting methods can lead to best results.

$$A_{T+j} = \beta P_{1,T+j} + (1-\beta) P_{2,T+j} + \varepsilon_{T+j} \tag{6}$$

Where A_{T+j} stands for actual value in $\mathrm{T+j}$ and $\mathrm{P}_{1,\mathrm{T+j}}$ is forecasted value of first method. Also $P_{2,T+j}$ is forecasted value of second method. For estimating β we use the following relation:

$$A_{T+j} - p_{2,T+j} = \beta(P_{1,T+j} - p_{2,T+j}) + \varepsilon_{c,T+j}$$

Estimation of β will give optimal combination of forecasts. The reason is sum up to 1 condition in forecast combination: $\alpha = 1-\beta, \alpha+\beta = 1$. For generalization to multimodel or multimethod forecasts, it can be written:

$$A_{T+j} = \beta_1 P_{1,T+j} + \beta_2 P_{2,T+j} + ... + \beta_k P_{k,T+j} + \varepsilon_{C,T+j}$$

As sum of weights for various forecasts should be one:

$$\beta_1 + \beta_2 + \cdots + \beta_k = 1$$

$$\beta_k = 1 - \beta_1 - \beta_2 + \cdots$$

$$A_{T+j} - p_{k,T+j} = \beta_1(P_{1,T+j} - P_{k,T+j}) + \beta_2(P_{2,T+j} - P_{k,T+j}) + \cdots + \beta_{k-1}(P_{k-1,T+j} - P_{k,T+j}) + \varepsilon_{c,T+j}$$

or in more compact form:

$$A^*{}_{T+j} = \beta_1 P_1^*{}_{T+J} + \beta_2 P_2^*{}_{T+j} + \cdots + \beta_{k-1} P^*{}_{K-1,T+j} + \varepsilon_{c,T+j}$$

Where all of the forecasts and actual values are stated as deviation from kth forecast value. Although this method is very useful it can not be applicable in estimating the size of underground economy. Because value of underground economy which is used as actual value is not measured accurately. As the main concern of this paper is a precise measurement and not a forecast of UE size, we use the second method i.e. variance-covariance method.

Variance- covariance tries to combine different forecasts or estimations to minimize variance of forecasts. This method relies on combination of forecast error of various methods. (Clements and Hendry 1998, p. 229)

$$e_{c,+T} = \alpha e_{1,t} + (1-\alpha)e_{2,t}$$

Where $e_{1,T+j}$ is forecast error of first method or model in time $\mathrm{T}+\mathrm{j}$ and $e_{2,T+j,T+j}$ is forecast error of the second method in the same time which combination of them is equivalent to combined error $e_{c,T+j}$. Minimizing variance equation leads to:

$$\mathrm{V}\left(\mathrm{e}_{\mathrm{c,t}}\right) = \alpha^2 \mathrm{V}\left(\mathrm{e}_{1,\mathrm{t}}\right) + \left(1-\alpha\right)^2 \mathrm{V}\left(\mathrm{e}_{2,\mathrm{t}}\right) + 2\alpha\left(1-\alpha\right)\mathrm{cov}\left(\mathrm{e}_{1,\mathrm{t}}, \mathrm{e}_{2,\mathrm{t}}\right)$$

Taking derivative respect to α

$$\alpha^* = \frac{V(e_{2,t}) - \mathrm{cov}(e_{1,t}, e_{2,t})}{V(e_{1,t}) + V(e_{2,t}) - 2\mathrm{cov}(e_{1,t}, e_{2t})} \qquad (7)$$

By putting α^* in prior equation, one can show that:

$$MSFE\left(P_{c,t}\right) \le \min\left\{MSFE\,(P_{1,t}),\ MSFE\,(P_{2,t})\right\}$$

Therefore with combination of forecasts in any way forecast error can be reduced. This method can be generalized and is applicable for various types:

$$e_{c,t} = \beta_1 e_{1,t} + \beta_2 e_{2,t} + \cdots + \beta_k e_{k,t}$$

$$V\left(e_{c,t}\right) = \sum_{j=1}^{k}\sum_{i=1}^{k}\beta_i\beta_j \,\mathrm{cov}\left(e_{i,t}, e_{j,t}\right)$$

As sum of weights $\left(\beta_1 + \beta_2 + \cdots + \beta_k\right)$ should be 1, optimal values of β_i is given by solving following restricted optimization problem:

$$L = \sum_{j=1}^{k}\sum_{i=1}^{k}\beta_i\beta_j \mathrm{Cov}\left(e_{i,t} e_{j,t}\right) + \lambda\left[\sum_{i=1}^{k}\beta_i - 1\right] \qquad (8)$$

A MATLAB code[1] is written by authors for solving the problem.

3- Data, Estimation, and Empirical Results

In the previous section literature of underground economy and common methods in estimation of UE size was presented. In this section, based on previous studies and making some adjustments because of data limitations in Iran we try to estimate the underground economy size.

1- The code is available through a formal request.

The data for estimation of models is the national accounts of 1959-2003(1338-1382 Iranian calendars) that are collected from various statistical sources such as Central Bank[1] of Iran, web site of Iran economy information[2] and Iran Statistics Center. First of all, a regression model based on liquidity demand is fitted and tax gap is calculated. Variables of the model are: $CASH$ as dependent variable and TAX/GDP, ratio of tax to gross domestic product, and $DIRECTAX/GDP$, ratio of direct tax to GDP, for modeling the effect of tax burden on liquidity demand, $RATE3$, one year bank deposits rate, for taking in to account the opportunity cost of cash demand and $PERINCCUR$, per capita income in current currency, for showing the effect of precautionary and transaction demand of money. In some models the ratio of $CASH/M_2$, ratio of cash to $M2$ definition of money, is used as a dependent variable. Also nominal balance of money is divided to CPI for converting to real balance. In this case, dependent variable of regression is denoted by $REALCASH$.

Based on previous studies various models are fitted and some results are presented in following sections.

$$LOG(CASH/M2) = -0.112+1.093LOG(CASH(-1)/M2(-1)) + 13.744LOG(1+DIRECTAX/GDP)- 0.086LOG(RATE3) + [MA(1)= -0.997,BACKCAST=1343]$$

$$F= 114.385 \qquad R^2=0.93$$

$$Pval.F=0.000000 \qquad DW=1.71$$

This model is one of cash demand models which was presented in detail in the second section. The dependent variable is the ratio of cash to $M2$ and independent variables are first lag of dependent variable, ratio of direct tax income to GDP and interest rate. As a matter of fact, various other variables are entered in model and by backward approach are deleted from model when they found to be insignificant. In this "general to specific" specification first type error level is considered to be 5 percent in all of cases. Complete results are presented in the appendix which show that the variables in model after deletion of insignificant variable are significant in 1

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percent level or less. Also model is overally significant that is shown by very low probability of F statistic. In addition diagnostic tests on residual of regression confirm no serial correlations for error term assumption and nonspurious regression. In order to achieve a proper model we used three different specifications for the liquidity model. In the first model (Appendix 1) both of tax variables (ratio of direct tax to GDP and ratio of indirect tax to GDP) were entered in the model. After the deletion of the indirect tax (this variable is not significant at 5 percent level) the direct tax ratio to GDP became significant at one percent level. This result is in line with theoretical considerations and empirical results, and confirm that direct tax have positive effect on liquidity demand and therefore on UE size. This result is interesting when it reveals that structure of tax (tax mix) can be important in changes of UE size.

In another model indirect tax ratio to GDP is replaced by ratio of total tax to GDP (Appendix: model 2). In the model there is multicolinearity between total tax ratio and direct tax ratio, when the indirect tax is a component of total tax. In the model, also the direct tax variable ratio is significant and total tax ratio variable is not significant.

Final model (Appendix model 3) is obtained by omitting the insignificant variable (total tax ration to GDP) from the model. Summary of the results of this model which is presented in above indicates that the model is significant in less than one percent level and signs of the variables are consistent with theory. Increase in tax causes increase in money demand and increase in bank deposits rate, reduces liquidity demand. Income variable has positive sign, but it is not significant in five percent level. Therefore, it is deleted from the model.

To estimate UE size from the regression mentioned above, tax burden term is deleted and the real balance of money in the absence of shadow economy (Theoretical reasons were presented before) is calculated:

LOG(CASH/M2) = -0.112+1.093LOG(CASH(-1)/M2(-1)) - 0.086LOG(RATE3)
+ [MA(1)= -0.997,BACKCAST=1343]

Then using the following equation size of underground economy is calculated:

$$UESIZE = (CASH - \exp(CASHf)M2)V2 / GDP \quad (9)$$

Where $V2$ is velocity of $M2$, $CASH$ is liquidity and $CASHf$ is predicted liquidity from final regression model and $UESIZE$ is the size of shadow economy that is calculated from the equation (9).

Table 1: Share of shadow economy and tax gap (Structural regression)

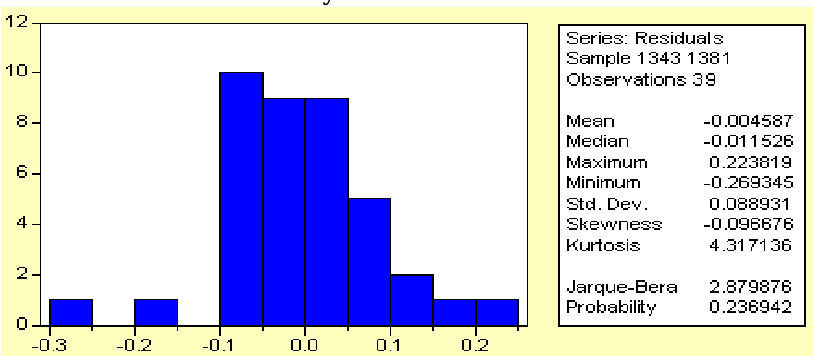
Year	Tax Gap (Billion Rials)	Shadow Economy's Share	Year	Tax Gap (Billion Rials)	Shadow Economy's Share
1963	1.51	7.31	1983	173.36	21.76
1964	2.03	8.76	1984	204.84	22.79
1965	2.77	9.7	1985	225.49	21.81
1966	3.59	10.34	1986	231.6	22.6
1967	4.27	10.95	1987	224.08	21.75
1968	5.38	10.82	1988	202.39	20.52
1969	6.49	10.72	1989	217.6	18.32
1970	7.47	10.58	1990	290.82	17.16
1971	8.26	10.05	1991	427.62	15.47
1972	10.55	10.28	1992	550.44	14.58
1973	14.06	10.72	1993	566.53	13.95
1974	17.09	10.83	1994	764.07	13.92
1975	31.38	11.59	1995	910.64	12.45
1976	37.75	11.01	1996	1417.23	11.28
1977	48.05	10.83	1997	1980.5	11.42
1978	122.32	26.26	1998	2183.07	11.68
1979	74.92	20.34	1999	2962.09	11.47
1980	73.5	21.59	2000	3714.39	10.09
1981	124.84	22.53	2001	3857.24	9.09
1982	135.11	22.01	2002	4176.02	8.33

Due to use of credit cards and automatic teller machines (ATM), it seems that velocity of the money have been increased in recent years. However, the hard currency in demand had little growth. This fact can be important in the estimation of UE size. Since there is no data about ATMs in Iran, we cannot reduce the bias of the estimation in our research.

Tax gap shows how much income tax could be gathered if the size of underground economy was zero. If one can shows that the main motive of underground activities is tax evasion, the policy recommendation is using indirect tax (especially excise tax) instead of income tax.

In order to examine validity of the estimated results and also to avoid the spurious regression we take Johanson cointegration test (Table 4 in appendices). The results of the test show that there is a long run relation between the ratio of cash to M2 as dependent variable and deposits rate and indirect tax to GDP ratio as independent variables. This relation is significant in less than one percent level of first type error. In other words, in addition to short run relation, there is a positive long run effect of direct tax to GDP ratio on size of underground economy.

Table 2: Normality Test for Residuals of the Model



The normality test of residuals shows that normality hypothesis can not be rejected at 5 percent level. Hence, in 5 percent level distribution of residuals is normal and test-statistics are valid. Also, based on following table one can concluded that there is no serial correlation.

Table 3. LM Test for Serial correlation

Breusch-Godfrey Serial Correlation LM Test:			
F-statistic	0.447325	Probability	0.643273
Obs*R-squared	0.957109	Probability	0.619678

In addition to the above mentioned models, other models are estimated, but because their results are not consistent with theory we do not report them in this paper. Some of these models are those used by Spiro (1996).

We tried to use MIMIC model, but because of covariance matrix error for Iran data, LISREL did not turn out any output for general model.

The Estimation of UE size by fuzzy approach is presented in the following table (this method is based on Gillis methods that was used for New Zealand). Although these methods underestimate the size of underground economy in some years they give average size of 16 percent for whole period which is acceptable and is consistent with other results such as regression method.

Table 4: Shadow economy's size and tax gap (Fuzzy logic Approach)

Year	Tax Gap (Billion Rials)	Shadow Economy's Share	Year	Tax Gap (Billion Rials)	Shadow Economy's Share
1963			1983	127.2011	15.97
1964	4.578288	19.73	1984	183.7123	20.442
1965	4.56742	15.97	1985	165.0819	15.97
1966	5.54159	15.97	1986	209.9815	20.494
1967	7.99149	20.49	1987	97.51358	9.4655
1968	9.757601	19.63	1988	85.68048	8.6853
1969	11.33765	18.71	1989	86.62404	7.2922
1970	14.52807	20.58	1990	270.6915	15.97
1971	13.12734	15.97	1991	549.4332	19.871
1972	20.73136	20.21	1992	751.0225	19.892
1973	9.879885	7.53	1993	648.5896	15.97
1974	10.74476	6.81	1994	876.8808	15.97
1975	54.35768	20.07	1995	1167.886	15.97
1976	72.18682	21.06	1996	2005.864	15.97
1977	87.74408	19.78	1997	3306.594	19.064
1978	98.53785	21.15	1998	3669.682	19.638
1979	75.1111	20.39	1999	5077.937	19.658
1980	45.38894	13.33	2000	7564.264	20.553
1981	48.29425	8.72	2001	7982.318	18.813
1982	99.62369	16.23	2002	4742.596	9.4585

To reduce forecast error one can use combination of two forecasts of liquidity demand model and fuzzy logic approach. Solving the following equations gives the share of each forecasting method (42.64 and 57.36 respectively) in combined forecasts.

$$L = \sum_{j=1}^{k} \sum_{i=1}^{k} \beta_i \beta_j Cov \left[e_{i,t} e_{j,t} \right] + \lambda \left[\sum_{i=1}^{k} \beta_i - 1 \right]$$

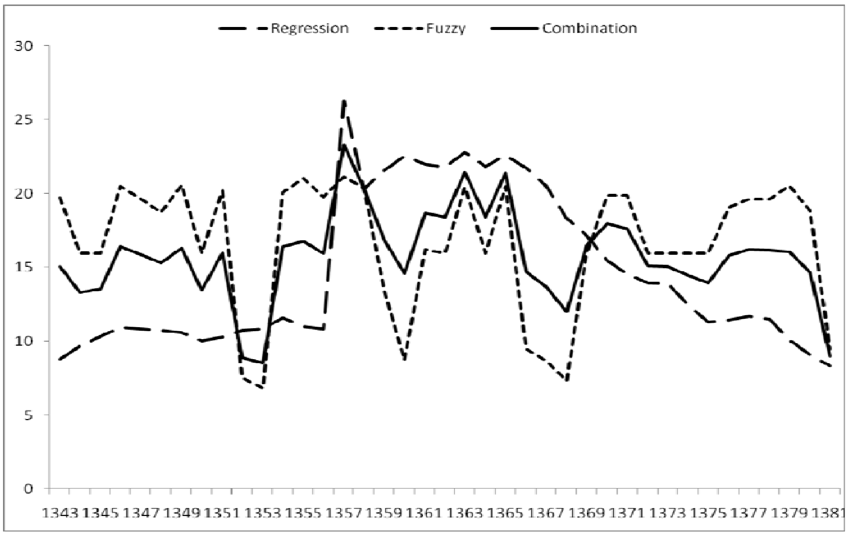


Figure 1: Combination of the Estimates of Liquidity Demand and Fuzzy logic Approach

Table 5: Combination of the Estimates of Liquidity Demand and Fuzzy logic Approach

Year	Regression	Fuzzy	Combined	Year	Regression	Fuzzy	Combined
1964	8.76	19.73	15.05446	1984	22.79	20.442	21.44456
1965	9.7	15.97	13.29485	1985	21.81	15.97	18.46191
1966	10.34	15.97	13.56783	1986	22.6	20.494	21.39353
1967	10.95	20.49	16.42214	1987	21.75	9.4655	14.70414
1968	10.82	19.63	15.8748	1988	20.52	8.6853	13.73002
1969	10.72	18.71	15.30123	1989	18.32	7.2922	11.99377
1970	10.58	20.58	16.31778	1990	17.16	15.97	16.47648
1971	10.05	15.97	13.44559	1991	15.47	19.871	17.99244
1972	10.28	20.21	15.97703	1992	14.58	19.892	17.6266

1973	10.72	7.53	8.889317	1993	13.95	15.97	15.10844
1974	10.83	6.81	8.52439	1994	13.92	15.97	15.09392
1975	11.59	20.07	16.45347	1995	12.45	15.97	14.47007
1976	11.01	21.06	16.77569	1996	11.28	15.97	13.97167
1977	10.83	19.78	15.96467	1997	11.42	19.064	15.80396
1978	26.26	21.15	23.32698	1998	11.68	19.638	16.24578
1979	20.34	20.39	20.36905	1999	11.47	19.658	16.16536
1980	21.59	13.33	16.85256	2000	10.09	20.553	16.09262
1981	22.53	8.72	14.60839	2001	9.09	18.813	14.66748
1982	22.01	16.23	18.69413	2002	8.33	9.4585	8.97668
1983	21.76	15.97	18.44084				

4- The relation between underground economy size and income taxes

Income taxes in most countries show the resource which is available for government in provision of utilities and gives a sense about government’s liberty in governance and developmental policies. When there is a negative effect of the UE size on the GDP and government's revenue and type of tax can be important in determination of UE size, we interested to study the relation between tax mix and UE size. In this paper we examine the relation of UE size and tax structure in a cross country framework. The question of this section is “what is the relation between tax mix and UE size?” Answer to this question gives us a better understanding of the possibility of reduction in UE size by changing tax mix. It is important to note that importance of the tax mix is not affected by the presence or absence of the relation between the underground economy and tax mix. Importance of the indirect tax is the availability of it even from underground economy activities. In other words, even when there is no empirical relation between direct tax and underground economy, because of the possibility of tax gathering indirectly from underground economy and impossibility of gathering income tax from this sector changing the tax mix is still recommended. One of the main motives of underground economy’s activities is income tax evasion. Therefore if income tax could be applicable (irrespective of possibility of the income tax gathering from this sector) for some of agents there is no motivation to enter to underground economy. Underground economy such as formal economy uses goods and services therefore there is no way for evasion from indirect tax and this not only leads extension of tax base, but it also shrinks UE size in future. The graphical illustration is as follows:

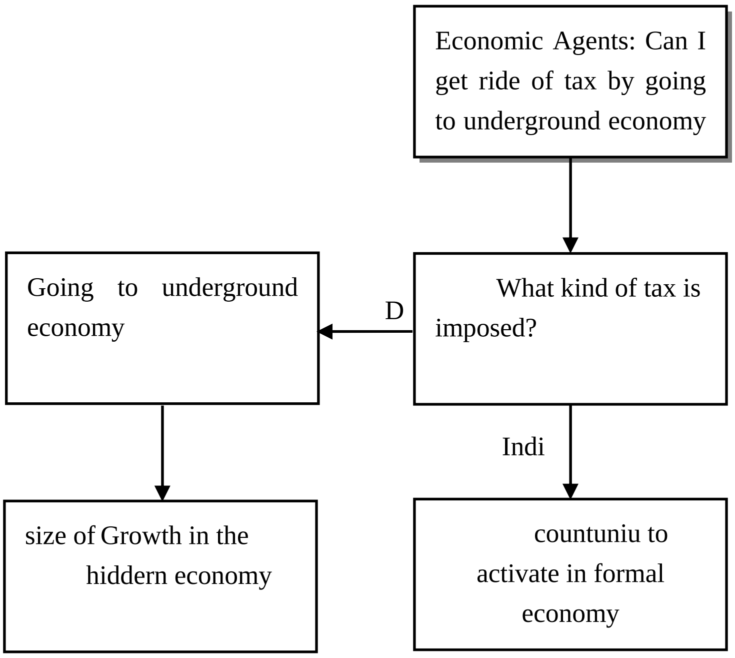


Figure 2: Behavior of Economic Agents in tax regimes

Therefore, changes in the tax mix can lead to change in share of informal economy in economy. Costs of increase in informal economy size are not solely economic costs, and social costs of this sector are one of the concerns in the most of the countries.

In most of the studies, panel data or cross section data is used for country comparisons. Time series data is not proper for the examining the relation between tax types and UE size, because of the estimated nature of UE size and also special conditions of Iran's economy. It may argue that estimated nature of the UE size may have bias effects in cross country studies. Although the argument is not incorrect, as the number of countries increases biases may cancel out each other and mean of them be less than the study that relies only on Iran's economy estimates. Therefore, in this paper we use cross country data for the examination of the tax mix and UE size.

As we know, no study has been done about relation between tax mix and UE size in Iran and also in other countries there is no study in panel data form. We use following form for our panel data approach in the study of the tax mix.

$$UE_{it} = \alpha + \beta_1 T_{1,it} + \beta_2 T_{2,it} + \ldots + \beta_k T_{K,it} + \varepsilon_{it} \qquad (10)$$

Where UE_{it} is the size of the underground economy for country i in time t, $T_{1,it}$ is the tax of the first type (income tax for example) for country i in time t, and $T_{k,it}$ is the share of the tax of the kth type for country i in time t. Because, only the information of 1996 is available t will be 1996.

We use following data for running a regression of tax mix and its relation to UE size. The estimated equation show that value added tax is not significant at 5 percent level and should be deleted from the regression.

Table 6. Data of informal sector size as percent of GDP and tax and Insurance burden on employers

Country	Informal Sector Size as Percent of GDP	Value added Tax (percent)	Direct Tax(Percewnt)	Social Secutrities Payments(employee)	Social Secutrities Payments(employer)	Social Secutrities Payments(Total)	Social Secutieties+Direct Tax	Social Secutieties+Direct Tax+Value Added Tax
Greec	28.5	18	11	15.8	27.5	43.3	54.3	72.3
Italy	27	19	12	9.9	32	41.9	53.9	72.9
Spain	22.9	16	13	6.6	31.6	38.2	51.2	67.2
Belgium	21.9	21	19	10	26	36	55	76
Sewden	19.2	25	20	4	29.6	33.6	53.6	78.6
Norway	18.9	23	19	7	12.8	19.8	38.8	61.8
Denmark	18.3	25	36	9	0	9	45	70
Ireland	15.9	21	20	7.2	12.3	19.5	39.5	60.5
Canada	14.6	7	21	7	8	15	36	43
Germany	14.5	15	18	16.1	16.1	32.2	50.2	65.2
France	14.3	20.6	6	13	31	44	50	70.6
Hetherlands	14	17.5	10	31	8.8	39.8	49.8	67.3
UK	13.1	17.5	16	10.7	10.2	21.4	37.4	54.9
USA	8.8	3	17	7.6	13.8	21.4	38.4	41.4
Austria	8.3	20	8	18.2	24.2	42.4	50.4	70.4
Switzerland	7.5	6.5	10	11.6	11.6	23.2	33.2	39.7

$$\hat{UE}_{it} = -7.54 + 0.09VAT + 0.62DIRETAX + 0.23SOCIAL1 + 0.54SOCIAL2$$

With strategy of general to specific, the insignificant variable SOCIAL1 also is deleted from the model. The final model is overally significant at 5 percent level.

$$\hat{UE}_{it} = -0.47 + 0.5DIRETAX + 0.49SOCIAL2$$
$$F = 5.69 \qquad Pvalue(F) = 0.0168$$
$$R^2 = 0.47$$

Also, Lagrange multiplier test for serial correlation shows that the model is acceptable in common levels of the significance. The above regression show that there is no relation between value added tax and UE size, whereas direct tax has significant positive effect on the UE size.

In the case of Iran, we can use M_2 as a proxy for underground economy size when it is highly correlated with underground economy size. Therefore the relation between UE size and tax variables can be studied via money aggregates such as M_2 .

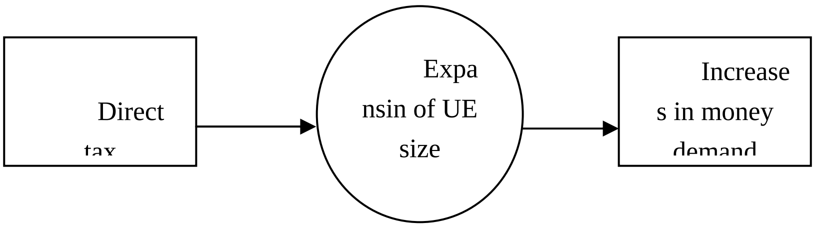


Figure 3: Relation between direct tax and UE size and money demand

As the t-statistic of the tax coefficient in the estimated regression is not significant can be concluded that indirect tax has no effect on underground economy size in Iran. But, the direct tax variable has significant positive relation with liquidity demand and therefore with the underground economy size.

Table 7: Results of the regression of the money demand on the direct tax, indirect tax, deposits rate and lags of the dependent variable

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-0.093101	0.141286	-0.658954	0.5145
LOG(CASH(-1)/M2(-1))	1.087081	0.034819	31.22072	0.0000
LOG(1+DIRECTAX/...	14.35383	2.000010	7.176878	0.0000
LOG(1+(TAX-DIREC...	-0.311677	1.409429	-0.221137	0.8263
LOG(RATE3)	-0.102976	0.092345	-1.115119	0.2729
MA(1)	-0.996930	0.070391	-14.16283	0.0000

Results for Iran are consistent with results of other studies. There is a positive and significant relation between direct tax and UE size. Also this study is in the line with (Bovi 2003, p. 65) that show a positive relation between direct tax and UE size by a correlation analysis for all types of tax.

5- Conclusion and policy prescriptions

Studies done in different economies show that almost in all of them part of economic activities is ignored intentionally or unintentionally from the economic statistics. Size of the underground economy for Iran is estimated about 18.5 percent by Schneider. Studies of Iranians also are in line with this estimate. However, studies in Iran give a range of 7 to 83 percent for UE size. All in all, the reasonable range for Iran's underground economy size is 12-25 percent. Previous studies give valuable information about Iran's underground economy size that were considered in this study. However, due to get update estimates and more accurate estimates two methods of the liquidity demand and fuzzy logic approach were used. Also, we combined the results of these two methods to reduce the error of the estimates (this feature of results were not in the previous studies). Also, we used the results for estimation of the tax gap and in determination of the tax mix and its relation with underground economy's size.

The results of our estimation for UE size is about 15 percent for recent years. Results of this study show high share of underground economy for special years such as 1957(the revolution of Iran) that confirm the robustness of the estimations for outlier observations.

The results of the forth section show there is a significant positive relation between the UE size and direct tax. Therefore changing the tax mix in favor of indirect tax can reduce the size of the underground economy and increase the attainable tax income for government. The reason behind of the indirect tax-dominated tax mix is that the indirect tax can be imposed on both of formal and informal activities and it reduces motivations of the economic agents to convey their activities in underground economy for tax evasion.

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Appendices

A1: Regression of the liquidity demand and all types of tax and rate

Dependent Variable: LOG(CASH/M2)
Method: Least Squares
Date: 07/28/05 Time: 02:33
Sample(adjusted): 1343 1381
Included observations: 39 after adjusting endpoints
Convergence achieved after 20 iterations
Backcast: 1342

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-0.111803	0.144073	-0.776016	0.4433
LOG(CASH(-1)/M2(-1))	1.091330	0.035357	30.86577	0.0000
LOG(1+DIRECTAX/...	14.18617	3.326054	4.265166	0.0002
LOG(1+TAX/GDP)	-0.112365	1.466619	-0.076615	0.9394
LOG(RATE3)	-0.090079	0.093735	-0.961003	0.3435
MA(1)	-0.997441	0.076890	-12.97239	0.0000

R-squared	0.931123	Mean dependent var	-1.731001
Adjusted R-squared	0.920687	S.D. dependent var	0.338599
S.E. of regression	0.095358	Akaike info criterion	-1.721720
Sum squared resid	0.300074	Schwarz criterion	-1.465787
Log likelihood	39.57353	F-statistic	89.22320
Durbin-Watson stat	1.727426	Prob(F-statistic)	0.000000

A2: Regression of the liquidity demand and direct tax and other variables

Dependent Variable: LOG(CASH/M2)
Method: Least Squares
Date: 07/28/05 Time: 02:35
Sample(adjusted): 1343 1381
Included observations: 39 after adjusting endpoints
Convergence achieved after 16 iterations
Backcast: 1342

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-0.112036	0.034117	-3.283892	0.0024
LOG(CASH(-1)/M2(-1))	1.092694	0.016500	66.22284	0.0000
LOG(1+DIRECTAX/...	13.74492	1.018073	13.50092	0.0000
LOG(RATE3)	-0.085640	0.017935	-4.775094	0.0000
MA(1)	-0.997373	0.058348	-17.09365	0.0000

R-squared	0.930830	Mean dependent var	-1.731001
Adjusted R-squared	0.922692	S.D. dependent var	0.338599
S.E. of regression	0.094145	Akaike info criterion	-1.768750
Sum squared resid	0.301352	Schwarz criterion	-1.555473
Log likelihood	39.49063	F-statistic	114.3851
Durbin-Watson stat	1.708103	Prob(F-statistic)	0.000000

A3: Long Run Relation between Liquidity demand and, tax and interest rate

Date: 07/29/05 Time: 00:16
Sample(adjusted): 1344 1383
Included observations: 40 after adjusting endpoints
Trend assumption: Linear deterministic trend
Series: LOG(CASH/M2) LOG(1+DIRECTAX/GDP) LOG(RATE3)
Lags interval (in first differences): 1 to 1

Unrestricted Cointegration Rank Test

Hypothesized No. of CE(s)	Eigenvalue	Trace Statistic	5 Percent Critical Value	1 Percent Critical Value
None **	0.555535	39.81795	29.68	35.65
At most 1	0.168501	7.382625	15.41	20.04
At most 2	4.08E-05	0.001631	3.76	6.65

*(**) denotes rejection of the hypothesis at the 5%(1%) level
Trace test indicates 1 cointegrating equation(s) at both 5% and 1% levels

A4. LM test full results for serial correlation test

Breusch-Godfrey Serial Correlation LM Test:

F-statistic	0.447325	Probability	0.643273
Obs*R-squared	0.957109	Probability	0.619678

Test Equation:
Dependent Variable: RESID
Method: Least Squares
Date: 07/29/05 Time: 01:22
Presample missing value lagged residuals set to zero.

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-0.004405	0.037362	-0.117892	0.9069
LOG(CASH(-1)/M2(-1))	0.001442	0.018802	0.076669	0.9394
LOG(1+DIRECTAX/...	0.262806	1.023930	0.256664	0.7991
LOG(RATE3)	-0.000871	0.018124	-0.048035	0.9620
MA(1)	-7.57E-05	0.000250	-0.303214	0.7637
RESID(-1)	0.125395	0.178602	0.702091	0.4877
RESID(-2)	0.036784	0.179546	0.204872	0.8390

R-squared	0.024541	Mean dependent var	-0.004587
Adjusted R-squared	-0.158357	S.D. dependent var	0.088931
S.E. of regression	0.095714	Akaike info criterion	-1.693760
Sum squared resid	0.293156	Schwarz criterion	-1.395172
Log likelihood	40.02833	F-statistic	0.134180
Durbin-Watson stat	1.972033	Prob(F-statistic)	0.990849

Table 1: Correlations between the Underground Economy and list determinants

RHS variables	Dependent variable: share of underground economy as a % official GDP — MODELS					
	A-general	A-specific	B-general	B-specific	C-general	C-specific
Corruption*	-0.74*** (-3.11)	-0.77*** (-3.65)	-0.79*** (-3.49)	-0.88*** (-4.11)	-0.79*** (-4.14)	-0.73*** (-4.05)
Rule of Law*	-0.43* (-1.79)	-0.44* (-1.85)	-0.55** (-2.22)	-0.56** (-2.22)	-0.60** (-2.61)	-0.58** (-2.44)
EPL*	0.02 (0.28)		0.068 (1.32)		0.05 (0.78)	
T1	0.34* (1.34)	0.38* (1.76)				
T2			0.21** (2.01)	0.20* (1.98)		
T3					0.20*** (4.19)	0.18*** (3.81)
T4					0.06 (0.78)	
T5					-0.07 (-1.35)	
Adj. R squared	0.28	0.29	0.30	0.32	0.38	0.38

Notes: *For Rule of Law and Corruption Indices, higher values mean 'better', vice versa for EPL (Employment Protection Legislation).
T1=Total tax as a % of GDP; T2=taxes on personal income as a % of GDP; T3=taxes on general consumption as a % of GDP;
T4=income tax as a % of labour costs; T5=employee and employer contributions as a % of labour costs.
*** Denotes significant at the 1% level; ** Denotes significant at the 5% level; * Denotes significant at the 10% level.
All variables are defined in logarithms; number of observations: 59; White's heteroscedastic-consistent t-statistics are in parentheses.
There are three models (A, B, C) according to the three different tax burdens (T1; T2; T3,4,5). Each regression is modelled, including all variables (general), and imposing some zero restrictions on the insignificant parameters (specific).

Source:

Bovi , Maurizio (2003).The Nature of the Underground Economy – Some Evidence from OECD Countries,JIIDT Vol. 7.

A5. Details of MIMIC Model

The MIMIC model can be presented in econometric specification. Let η be an unobserved scalar variable (Underground variable which measures size of under ground economy), $Y' = (y_1, y_2, ..., y_P)$ vector of indicators of η and $X' = (x_1, x_2, ..., x_q)$ vector of causes for η. Also suppose $\lambda_{(p\times1)}$ and $\gamma_{(q\times1)}$ as vectors of parameters, and $\in_{(p\times1)}$ and ζ as error term. All of the Elements of $\in$ and ζ are normally distributed and,

$Var(\zeta) = \psi \quad Cor(\in) = \theta_\varepsilon$.

Then the MIMC model can be stated as

$y = \lambda\eta + \varepsilon$ (a1)

$\eta = \gamma'\alpha + \zeta$ (a2)

By replacement of $(a2)$ in $(a1)$, the model can be written as $y = \pi x + z$, where $\pi = \lambda\gamma'$. Also $z = \lambda\zeta + \in$ and $Cov(z) = \lambda\lambda'\psi + \theta_{\varepsilon}$, since $\in$ and ζ are independently distributed. Rank of matrix of regressors is one and error matrix is constrained; numerical estimation of parameters is not possible. Estimates are function of parameters and represent relative importance of parameters. Then, normalization of parameters is required.

Since x and y are given, the equation can be estimated by maximum likelihood method. By estimated value of π one can get $\gamma = \lambda$. Putting $\zeta = \mathrm{E}(\zeta) = 0$ we can forecast η and UE size. This estimated value for UE size is an ordinal value in order to reach the cardinal value at least in one year UE size should be determined cardinally by other methods. This fact is the main limitation of this method.

One of conventions in analysis of underground variables is LISREL[1] method. In this method two types of observable variables are $X = (x_1, x_2, \ldots, x_q)', Y = (y_1, y_2 \ldots, y_p)'$ which are in deviation from mean form. It is supposed that variables X, Y satisfy the conditions of factor analysis models. Common factores can be represented by $\eta = (\eta_1, \eta_2, \ldots \eta_m)'$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_n)'$, and unique factors $\delta = (\delta_1, \delta_2, \ldots \delta_q)'$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p)'$. So:

$$Y = \Lambda_Y \eta + \varepsilon$$

$$X = \Lambda_X \xi + \delta$$

Where Λ_x, Λ_y are loading matrixes with q_{xn}, p_{xm} dimensions. Traditional assumptions of factor analysis are satisfied in our problem

$$E(\eta) = 0,\ E(\xi) = 0,\ E(\varepsilon) = 0,\ E(\delta) = 0$$

$$E(\eta\varepsilon') = 0,\ E(\xi\delta') = 0,\ E(\varepsilon\varepsilon') = \Theta^2_{\varepsilon},\ E(\delta\delta') = \Theta^2_{\delta},\ E(\varepsilon\delta') = 0$$

Where $\Theta_{\delta}, \Theta_{\varepsilon}$ are diagonal matrices. Factors ξ, η are correlated in whithin and between sets. In order to estimate parameters let Ω be covariance matrix of $(\eta', \xi')'$:

1- Linear Interdependent Structural Relationship

$$\Omega = \begin{pmatrix} \Omega_{\eta\eta} & \Omega_{\eta\xi} \\ \Omega_{\xi\eta} & \Omega_{\xi\xi} \end{pmatrix}$$

$$\Omega_{\eta\eta} = B^{-1}\Gamma\Phi\Gamma' B'^{-1} + B^{-1}\Psi B'^{-1}$$

$$\Omega_{\eta\xi} = \Omega'_{\xi\eta} = B^{-1}\Gamma\Phi$$

$$\Omega = \Phi$$

The model can be estimated by maximum likelihood method. Variance-covariance matrix of (y', x') is as follows:

$$\Sigma = \begin{pmatrix} \Sigma_{yy} & \Sigma_{yx} \\ \Sigma_{xy} & \Sigma_{xx} \end{pmatrix}$$

Where

$$\Sigma_{yy} = \Lambda_y \Omega_{\eta\eta} \Lambda'_y + \Theta_4^2$$

$$\Sigma_{xy} = \Sigma'_{xy} = \Lambda_y \Omega_{\eta\xi} \Lambda'_x$$

$$\Sigma_{xx} = \Lambda_x \Omega_{\xi\xi} \Lambda'_x + \Theta^2{}_{\delta}$$