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GJR-Copula-CVaR Model for Portfolio Optimization: Evidence for Emerging Stock Markets

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Abstract

This paper empirically examines the impact of dependence structure between the assets on the portfolio optimization, composed of Tehran Stock Exchange Price Index and Borsa Istanbul 100 Index. In this regard, the method of the Copula family functions is proposed as powerful and flexible tool to determine the structure of dependence. Finally, the impact of the dependence structure on the risk identification and the optimized portfolio selection, will be analyzed. The results show that the t-student copula function provides the best performance among other Copula functions. Also, empirical evidence suggests that the performance of the GJR-Copula-CVaR method is relatively more accurate and more flexible than other common methods of optimization.

Keywords: Portfolio Optimization, Conditional Value at Risk, Copula Functions, Dependence Structure.
JEL Classification: C60, C61, G11.

1. Introduction

Modern Portfolio Theory (MPT) argues that investment is not only the selection of assets but selected a diversified combination of assets. One of the most important factors in portfolio diversification, dependency structure between the return of assets on the portfolio. According assuming normal distribution of asset returns in Portfolio Modern Theory, dependence between asset returns is defined perfectly by the linear correlation coefficients; that is, the variance of the risky assets portfolio only depends on the variance of each individual assets constituent portfolio and the linear correlation between the assets. But in the literature of past two decades, a lot of evidence has been provided about the breaches of the assumption of normal distribution (Harvey

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and Siddique, 1999; Sadique and Silvapulle, 2001; Hartman et al., 2004; Poon et al., 2004; Dass and Uppal, 2004). So, the linear correlation coefficients will not be an appropriate criterion in stating the dependence structure between assets return and it can lead to wrong results.

In order to fix this issue, the different solutions have been provided in the literature. Harvey and Siddique (1999) and Brooks et al. (2005) offer m-GARCH model with conditional skewness. Ang and Bekaert (2002) also benefited Markov Switching models for modeling the dependence structure. Despite the ability of these models to consider for time-varying conditional correlation, they cannot reproduce asymmetries in tail dependence.[1] To reproduce the asymmetric dependence Hartmann et al. (2004), Poon et al. (2004) and Beine et al. (2010) have used the extreme value theory. The major subject of this model is the problem of selecting the appropriate threshold and extreme value distribution. In order to consider the asymmetric dependence structure, Fantazzini (2008) proposes the dependency structure calculated based on Kendall's tau instead of linear correlation.

In financial literature, an alternative approach has been suggested based on Copula theory. Copulas theory are used to describe the dependence structure based on the multivariate joint distribution. Patton (2006) applied conditional copula model for determining the joint distribution of daily exchange rates and he found that the dependence structure of exchange rate is asymmetric. Palaro and Hotta (2006), for eliminated the problem of linear correlation coefficient, identified multivariate distributions of two US stock market index by the conditional copula and showed how conditional copula theory can be a very powerful tool in estimating the portfolio's Value at Risk (VaR). He and Gong (2009) constructed a copula- Conditional Value-at-Risk (CVaR) model for credit risks of listed company on Chinese security market and this model can exactly measure the coupled risks in financial market. Huang et al. (2009), Chen and Tu (2013) and Boubaker and Sghaier (2013), for omitting the limitation in the financial assets joint distribution included in portfolio which will result to the error estimation in VaR,

[1]. Tail dependence refers to dependence that arises between random variables from extreme observations.

used the conditional Copula- GARCH model and also indicted that because of the flexibility specification in distribution, the model is more appropriate for the financial markets high volatility study. Also, to estimate the risk of portfolio more accurately, Deng and et al. (2011) combine pair Copula-GARCH with the extreme value theory. They indicate that the optimal portfolio is better via pair Copula-GARCH-EVT model than other conventional model.

In this paper, I initially focus on modeling the dependence structure between the assets returns using copula functions. So that, the dependence structure between the assets returns that estimated by copula functions, will be replaced linear correlation coefficients. As asset, a pair of daily stock returns is considered. Therefore, to replace pattern of tail behavior rather than the joint multivariate normal distribution, GARCH and GIR models have been combined with Copula function. Finally, to investigate the effect of the dependence structure on the optimal portfolio, VaR and CVaR methods are applied. Thus, in addition to studying the effect of the dependence structure, it is possible to compare the accuracy of proposed models in portfolio risk estimation. Our findings support the issue that due to the flexibility and accuracy of distribution, the GJR-Copula-CVaR method is relatively more accurate and more appropriate than other common methods of optimization for portfolio risk identification and portfolio optimization.

In the next section, the CVaR GIR-Copula method, including the marginal distribution, dependence structure modeling and the CVaR estimation presented. In section 3, we will examine the effect of the dependence structure on portfolio optimization. In section 4, I carry out empirical analysis. In the final section, the conclusion will be expressed.

2. Method

In this section, we present the methodology of copula-GARCH-CVaR model by three steps. In first step, we propose the GARCH and GJR model to estimation the marginal distribution. In second step, we have introduced the copula functions and modeling of the dependence structure between assets returns is discussed. In the last step, we will pay on portfolio optimization based on CVaR approach with the dependence structure by copula functions.

Step 1: The marginal distribution

The conditional variance and average rate of returns estimation will be required for Value at Risk calculation. In some cases, the returns conditional distribution has the conditional heteroscedasticity, so paying attention to this specification, will cause to access to the estimators of efficient conditional maximum likelihood. On the other hand, it can be seen in some cases that the residual component faces with the heteroscedasticity variance effects' problem, i.e. the conditional auto-correlated variance. So we use the GARCH model and GJR models for the marginal distribution estimation, as powerful tools in time series data. Engle (1982), first used these models for estimation of financial time series, such as stock returns and exchange rate. The GARCH model is as follows:

$$r_t = \Phi_0 + \sum_{i=1}^{p} \Phi_i r_{t-i} - \sum_{i=1}^{q} \theta_i a_{t-i} + a_t \quad ; \quad a_t = \sigma_t \varepsilon_t \tag{1}$$

$$\sigma_t^2 = \alpha_o + \sum_{i=1}^{m} \alpha_i a_{t-i}^2 + \sum_{j=1}^{n} \beta_j \sigma_{t-j}^2$$

$$\varepsilon_t =\sim N(0.1) \ or \ \sim t_d.$$

where, $\{\varepsilon_t\}$ Is the sequence of non-correlation random variables with a distribution with a mean of 0 and a variance of 1 and furthermore we have: $\sum_{i=1}^{\max(m.n)}(\alpha_i + \beta_i) < 1,\ \beta_j > 0\ , \alpha_i \geq 0,\ \alpha_0 > 0$. It is simply understandable that for $i > m$, always we have $\alpha_i = 0$ and for $j > n$, $\beta_j = 0$. In addition to the d are the degrees of freedom. The applied method of estimating in parameters will be the MLE method. Based on the arguments Huang et al. (2009) assuming that $\Omega_{t-1} = \{a_0. a_1. \dots. a_{t-1}\}$, by given data $a_1. \dots. a_t$, the log- likelihood that can create the maximized numerically based on the MLE method, follows the (3) equation:

$$LLF = \sum_{k=0}^{n-1} f(a_{n-k}|\Omega_{n-k-1}) \tag{3}$$

The condition can be used for each ε_t distribution for the volatility modeling. By using the $(x_1.x_2.\dots.x_t)$ variables, can define the X_{t+1}'s conditional marginal distribution for GARCH model as follows:

$$\begin{aligned}(X_{t+1} \le x|\Omega_t) &= P(a_{t+1} \le (x-\mu)|\Omega_t) \\ &= P\left(\varepsilon_{t+1} \le \frac{(x-\mu)}{\sqrt{\alpha_0 + \alpha_1 a_t^2 + \beta\sigma_t^2}}|\Omega_t\right) \\ &= \begin{cases} N\left(\frac{(x-\mu)}{\sqrt{\alpha_0 + \alpha_1 a_t^2 + \beta\sigma_t^2}}|\Omega_t\right). & if\ \varepsilon \sim N(0,1) \\ t_d\left(\frac{(x-\mu)}{\sqrt{\alpha_0 + \alpha_1 a_t^2 + \beta\sigma_t^2}}|\Omega_t\right). & if\ \sim t_d \end{cases}\end{aligned} \tag{4}$$

The fundamental GARCH model's main problem is that it considers the variance process as symmetric. However, the studies suggest that the negative shock is likely to cause more volatility than the positive shock which is attributed to the leverage effects on the financial return. Therefore we also use the GIR model by Glosten et al. (1993) that is defined as (5) equation:

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i a_{t-i}^2 + \sum_{j=1}^{n} \beta_j \sigma_{t-j}^2 + \sum_{j=1}^{n} \gamma\, s_{t-j} a_{t-j}^2 \tag{5}$$

where $s_{t-1}\begin{cases}1. & a_{t-1} < 0 \\ 0. & a_{t-1} \ge 0\end{cases}$

moreover,

$\alpha_0 > 0,\ \alpha_1 > 0,\ \beta \ge 0,\ \beta + \gamma \ge 0$ and $\alpha_1 + \beta + \frac{1}{2}\gamma \le 1.$

also s_t is a dummy variable with the values of 1 and 0 when ε_t is negative and positive respectively. In GIR model, asymmetric effects will result in γ coefficients, when γ will be positive; that is the negative shocks represent more volatile than the positive shocks of the same size and same period. To follow the Huang et al. (2009), the X_{t+1}'s conditional marginal distribution in GJR model will be as follows:

$$P(X_{t+1} \leq x|\Omega_t) = P\left(\varepsilon_{t+1} \leq \frac{(x-\mu)}{\sqrt{\alpha_0+\alpha_1 a_t^2+\beta\sigma_t^2+\gamma s_t\varepsilon_t^2}}|\Omega_t\right) = \begin{cases} N\left(\frac{(x-\mu)}{\sqrt{\alpha_0+\alpha_1 a_t^2+\beta\sigma_t^2+\gamma s_t\varepsilon_t^2}}|\Omega_t\right), & if\ \varepsilon \sim N(0,1) \\ t_d\left(\frac{(x-\mu)}{\sqrt{\alpha_0+\alpha_1 a_t^2+\beta\sigma_t^2+\gamma s_t\varepsilon_t^2}}|\Omega_t\right), & if\ \sim t_d \end{cases} \tag{6}$$

Step 2: dependence structure modeling

Our idea in this paper is that we use the new dependence structure instead of the linear correlation coefficient in portfolio optimization. To do this we used Copula functions. Sklar (1959) stated the Copula functions based on this theorem for the first time, which indicated if $F(x_1.x_2)$ was an indicative of a joint distribution function for two-dimensional random vector X_1 and X_2 with a marginal distribution of $F_1(x_1)$ and $F_2(x_2)$, so a Copula function like C for all real values of $x_1$and x_2 would be as equations (7):

$$F(x_1.x_2) = C\big(F_1(x_1).F_2(x_2)\big) \tag{7}$$

Which by differentiation of the equation (7) both sides:

$$\frac{\partial^2 F(x_1.x_2)}{\partial x_1 \partial x_2} = \frac{\partial^2 C\big(F_1(x_1).F_2(x_2)\big)}{\partial F_1 \partial F_2} f(x_1)f(x_2) = \frac{\partial C(u_1.u_2)}{\partial u_1 \partial u_2} \times \prod_i \frac{\partial F(x_i)}{\partial x_i} = C(\tilde{u}) \times \prod_i f_i(x_i) \tag{8}$$

As f_i is the density function of F_i and $u_i = F_i(x_i)$ for each $i = 1.2$ and $\tilde{u} = (u_1.u_2)$ and $C(\tilde{u})$ is the copula density function. In the case of continuous variables, Sklar's theorem [1959] indicates that each multivariate probability distribution function can be defined as equation (8) with a marginal distribution and dependence structure. If all the margins are continuous, thus the Copula will be unique and can be defined by different values of the marginal distribution functions uniquely. The result main specification is that there is no necessity for the marginal distribution similarity; and it is not necessary the Copula selection is confined to the marginal distribution. Now we can define the linear correlation coefficient, as an indication of the dependence structure, by a kind of dependence by name of tail dependence and based on the F distribution function. The tail dependence, measures the dependence between the variables in the upper and lower quartiles on$I^2 = [0.1]$, which are defined as the upper and lower tails dependence. The Copula family, has been used here, including Gaussian Copula, t-Student Copula and Archimedean Copulas are as follows:

- Gaussian (Normal) copula

Song (2000), expressed the normal copula family distribution function as equation (20):

$$C^{Ga}(u.\nu;\rho) = \quad _{\rho}\left(\psi^{-1}(u).\psi^{-1}(\nu)\right) \tag{20}$$

That ψ_ρ is the multivariate standard normal distribution function with the correlation coefficient of $\rho \in (0.1)$; considering the equation (20), the joint distribution function and the joint density copula function of the family will be respectively as :

$$C^{Ga}(u.\nu.\rho) = \int_{-\infty}^{\psi^{-1}(u)} \int_{-\infty}^{\psi^{-1}(\nu)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{\frac{2u\nu - u^2 - \nu^2}{2(1-\rho^2)}\right\} dud\nu \tag{21}$$

$$C^{Ga}(u.\nu.\rho) = \frac{1}{\sqrt{1-\rho^2}} \exp\left\{\frac{u^2+\nu^2}{2} + \frac{2u\nu - u^2 - \nu^2}{1-\rho^2}\right\}$$

- t-student copula

Embrechts et al. (2002) expressed t-student copula distribution function as equation (19):

$$C^{T}(u.v) = T_{v.\rho}\left(t_v^{-1}(u).t_v^{-1}(v)\right) = \int_{-\infty}^{t_v^{-1}(u)}\int_{-\infty}^{t_v^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}}\left(1+\frac{s^2+t^2-stp}{v(1-\rho^2)}\right)^{-\frac{v+2}{2}} ds\,dt \quad (19)$$

Here $T_{v.\rho}$ is multivariate t- student distribution , ρ is the correlation coefficient, v is the degrees of freedom and t_v^{-1} is the inverse t-student distribution; they showed that the t-student copula function expressing the upper and lower tails dependence at the same time.

- Archimedean Copulas

Archimedean Copula, is an important category of Copula functions with simple structure and many analytical features. The bivariate Archimedean Copula, is $(u.v) = \psi^{[-1]}\{\psi(u) + \psi(v)\}$; that is a continuous strictly decreasing convex function is ψ: $[0.1] \to [0.\infty]$, as $\psi(1) = 0$ and the pseudo inverse function $\psi^{[-1]}$ is as the equation (9):

$$\psi^{[-1]}(t) = \begin{cases} \psi^{-1}(t) & 0 \le t \le \psi(0) \\ 0 & \psi(0) \le t \le \infty \end{cases} \quad (9)$$

If $\psi(0) = \infty$ will be strictly.(Schmidt (2003)). The general relationship between τ_c and the generator of Archimedean Copula is stated as equation (10):

$$\tau_c = 1 + 4\int_0^1 \frac{\psi(t)}{\acute{\psi}(t)}\, dt \quad (10)$$

The dependence coefficients of the upper tail dependence (λ_U), and the dependence coefficient of the lower tail dependence (λ_L), can be expressed respectively in terms of the generate functions.

$$\lambda_U = 2 - 2\lim_{s \to 0} \frac{\acute{\psi}^{-1}(2s)}{\acute{\psi}^{-1}(s)} \tag{11}$$

$$\lambda_L = 2\lim_{s \to \infty} \frac{\acute{\psi}^{-1}(2s)}{\acute{\psi}^{-1}(s)} \tag{12}$$

There are three common kinds of Archimedean Copula including Clayton Copula (Clayton (1978)), Frank Copula (Frank (1979)) and Gumbel Copula (Gumbel (1960)).

Clayton copula function has an asymmetric distribution, as in which the dependence on the negative tail is more than the positive one.

$$C_c(v_1.v_2) = \max\left[\left(v_1^{-\theta} + v_2^{-\theta} - 1\right)^{-\theta^{-1}}.0\right] \tag{13}$$

That its generator function is as equation (14):

$$\psi(t)\theta^{-1}\left(t^{-\theta} - 1\right). \quad \text{where } \theta \in [-1.+\infty) \tag{14}$$

where the upper tail dependence is equal to zero ($\lambda_{U_c} = 0$) and the lower one will be$\lambda_{L_c} = 2^{-\theta^{-1}}$.

Gumbel copula function has an asymmetric distribution as the Clayton copula; and unlike the former function, the dependence on the positive tail is more than the negative one.

$$C_G(v_1.v_2) = \exp\left(-\left[(-\ln(v_2))^{\theta} + (-\ln(v_2))^{\theta}\right]^{\theta^{-1}}\right) \tag{15}$$

This copula function will also have the equation (16)’s generator function.

$$\psi(t) = (-\ln(t))^{\theta}. \quad \text{where } \theta \geq 1 \tag{16}$$

Thus, in Gumbel copula $\lambda_{U_G} = 2 - 2^{-\theta}and\lambda_{L_G} = 0$.

Frank copula function indicates symmetric condition of Archimedean copula as equation (17):

$$C_F(v_1 v_2) = -\frac{1}{\theta}\ln\left(1 + \frac{\left(e^{-\theta v_1} - 1\right) + e^{-\theta v_2} - 1}{e^{-\theta} - 1}\right) \qquad (17)$$

with the generator function of:

$$\psi(t) = -\ln\left(\frac{e^{-\theta t} - 1}{e^{-\theta} - 1}\right) \quad . \ \theta \neq 1 \qquad (18)$$

According to the Frank copula function symmetry in this copula function, the upper tail dependence will be zero ($\lambda_{U_c} = 0$) and the lower one is zero too ($\lambda_{L_F} = 0$).

This study will benefit from the Maximum Likelihood Estimation or MLE method for estimating the copula parameters. It is assumed that the $X_1. \dots . X_n$ are the random variables with the distribution functions of $F_1. \dots . F_n$ and the relative parameters of the distributions are respectively $\alpha_1. \dots . \alpha_n$, with the joint distribution function of H, so considering the Sklar's theorem, have:

$$H(x_1. \dots . x_n | \alpha_1. \dots . \alpha_n. \theta) = C\left(F(x_1). \dots . F(x_n)\right) \qquad (22)$$

where C is the copula function with θ parameter.
And the $X_1. \dots . X_n$ joint density function is as equation (23):

$$h(x_1. \dots . x_n | \alpha_1. \dots . \alpha_n. \theta) = c(F(x_1; \alpha_n). \dots . F(x_n; \alpha_n); \theta) \prod_{i=1}^{n} f_i\left(x_i; \alpha_j\right) \qquad (23)$$

where c is the C copula function's density function. Thus, the log-likelihood function is as the equation (24):

$$l(\alpha_1.\dots.\alpha_n.\theta) = \prod_{t=1}^{n}\big(c(F_1(x_1;\alpha_n).\dots.F_n(x_n;\alpha_n);\theta)\big)\prod_{i=1}^{n} f_i\big(x_i;\alpha_j\big) \tag{24}$$

That is:

$$\ln\big(l(\alpha_1.\dots.\alpha_n.\theta)\big) = \sum_{t=1}^{n}\ln c(F_1(x_1;\alpha_n).\dots.F_n(x_n;\alpha_n);\theta) + \sum_{t=1}^{N}\ln\Big(f_i\big(x_i;\alpha_j\big)\Big) \tag{25}$$

The equation (25) differentiation, in relation to each $\alpha_1.\dots.\alpha_n$ and θ parameters will be calculated, then by uniting the resulting equations to zero, the parameters estimated. Furthermore, for selecting an appropriate copula function, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Log-likelihood Function are used for data processing.

Step 3: Conditional Value at Risk Estimation

In this paper, in order to achieve accurate concept of risk, rather than using the variance approach, Conditional Value at Risk (CVaR) approach used to measure risk of portfolio. The portfolio optimization based on CVaR minimizing with considering the dependence structure by Coppola functions, will be modeled as equation (26).

$$\min_{x.w.LP} \quad CVaR(\alpha) = \overline{\omega} + \frac{1}{(1-\alpha)}\sum_{t=1}^{T-1} LP_t \tag{26}$$

$$\text{subject to: } \; LP_t \geq \sum_{i=1}^{3}(L_{t-1.i} - P_{t.i})\, x_i - \overline{\omega} \;,$$

$$t = 1, \dots, T$$

$$\sum_{i=1}^{3} r_i w_i \geq G$$

where i indicates the portfolio's comprising assets. The $= 1, \dots, T$, is the option that indicates the different scenarios considering the intended period. G indicates the portfolios minimum average return. $P_i = P_{0.i}$, is the opening price; $P_{i.t}$ is the general element of the matrix which concludes the stock price in different conditions; r_i is the ith asset's expected return; $\overline{\omega}$ is the approximately variable of the portfolio's Value at Risk at the α confidence level. $LP_t(t = 1, \dots, T)$ Represents the portfolios loss which exceeds α threshold in different scenarios.

Now we can clarify the CVaR bounds. Following Mesfioui and Quessy [2005], for determining the VaR bounds of total n dependence risk $S = \sum_1^n X_i$ at the confidence level α percent, in which both first and second moments X_i are $\sigma_i^2 = \mathrm{V}(X_i) > 0$ and $\mu_i = E(X_i) > 0$ and each X_i marginal distribution is defined by (F_i). But the $(X_1 . \dots . X_n)$dependence structure is modeled by the unknown copula of $C\big(F_1(X_1) . \dots . F_n(X_n)\big) = F(X_1 . \dots . X_n)$.

$$\underline{Var}_{C_U}(\alpha) \leq Var_\alpha(S) \leq \overline{Var}_{C_L}(\alpha) \tag{27}$$

$\underline{Var}_{C_U}(\alpha)$ is the upper bound's clarification according to definition (28):

$$\underline{Var}_{C_U}(\alpha) = \sup_{C_L(u^{\setminus n}.1) \leq \alpha} \left[\sum_{i=1}^{n-1} F_i^{-1}(u_i) + F_n^{-1}\{C_{U.u^{\setminus n}}^{-1}(\alpha)\} \right] \tag{28}$$

where $C_U = \min(u_1 . \dots . u_n)$ will be $u^{\setminus n} = (u_1 . \dots . u_{n-1})$ and Fréchet - Hoeffding's lower bound.

$\overline{Var}_{C_L}(\alpha)$ will also be the upper bound clarification according to definition (29):

$$\overline{Var}_{C_L}(\alpha) = \inf_{C_L(u^{\setminus n}.1)\leq\alpha}\left[\sum_{i=1}^{n-1} F_i^{-1}(u_i) + F_n^{-1}\{C_{U.u^{\setminus n}}^{-1}(\alpha)\}\right] \quad (29)$$

where, $C_L = \max(\sum_{i=1}^{n} u_i - n + 1.0)$ is the Fréchet-Hoeffding's upper bound.

If it is assumed that the F_i marginal distribution is unknown in definition (28), then this definition will rewrite as definition (30):

$$\underline{Var^*}_{C_U}(\alpha) \leq Var_\alpha(S) \leq \overline{Var^*}_{C_L}(\alpha) \quad (30)$$

$\underline{Var^*}_{C_U}$ will be the upper bound clarification with definition (31):

$$\underline{Var^*}_{C_U}(\alpha) = \sup_{C_L(u^{\setminus n}.1)\leq\alpha}\left[\sum_{i=1}^{n-1} g_{\mu_i\sigma_i}(u_i) + g_{\mu_i\sigma_i}\{C_{U.u^{\setminus n}}^{-1}(\alpha)\}\right] \quad (31)$$

Where $C_U = \min(u_1.\dots.u_n)$ will be $g_{a.b}(n) = \{a - bq(1 - u)\}1_{\left(u\geq\frac{b^2}{a^2+b^2}\right)}$ and $u^{\setminus n} = (u_1.\dots.u_{n-1})$ Fréchet -Hoeffding's lower bound.

$\overline{Var^*}_{C_L}(\alpha)$ is also the upper bounds clarification by the definition (32):

$$\overline{Var^*}_{C_L}(\alpha) = \inf_{C_L(u^{\setminus n}.1)\leq\alpha}\left[\sum_{i=1}^{n-1} h_{\mu_i\sigma_i}(u_i) + h_{\mu_i\sigma_i}\{C_{U.u^{\setminus n}}^{-1}(\alpha)\}\right] \quad (32)$$

where $h_{a.b}(u) = a - bq^2(u)1_{\left(u \geq \frac{b^2}{a^2+b^2}\right)} + bq(u)1_{\left(u \geq \frac{b^2}{a^2+b^2}\right)}$ and $q(u) = \sqrt{u/(1-u)}$ is a strictly increasing function on an interval$[0.1]$.

3. Empirical Results

3.1 Data

The data of this analysis consists Tehran Stock Exchange Price Index (TEPIX) and Borsa Istanbul 100 Index (BIST 100) for the period November 2008-April 2015. TEPIX data are obtained from TSE database and BIST 100 from Yahoo Finance database. Stock market return have been calculated in logarithm form. Since only three days of trading days of the week on Tehran and Istanbul stock exchange overlapped, the frequency of our data is three days. Thus, our final sample consisted 2034 daily observations. Table 1 summarizes sample descriptive statistics of the each return series.

Table 1: Summary Statistics

Series	TEPIX	BIST 100
Mean	-0.0003	0.0041
Max	1.2527	1.6486
Min	-6.8916	-2.4849
Standard deviation	0.2321	0.1431
Skewness	-25.8041	-3.5803
Kurtosis	7.3098	17.9569
Jarque–Bera	3889.84***	5631.56***
ADF	-8.51***	-32.06***
R/S	1.21	1.02

***A rejection of the null hypothesis at the 1% significance level.

It’s understandable that kurtosis coefficient of the series returns is more than normal density function coefficient of kurtosis. All returns are skewed and The Jarque-Bera statistic denotes that the variables’ return distribution function is not normal and this lack of normality is a justification for copula functions’ using. Also, Augmented Dickey–Fuller (ADF) statistic show that the series are stationarity. The R/S test results also indicate the lack of a long-term memory among the used data; because the value of this statistic, by subtracting number 1, will be a number between the domains of (-0.5, 0.5). It can also be easily seen in Figure 1, where it is clear that have fat tails.

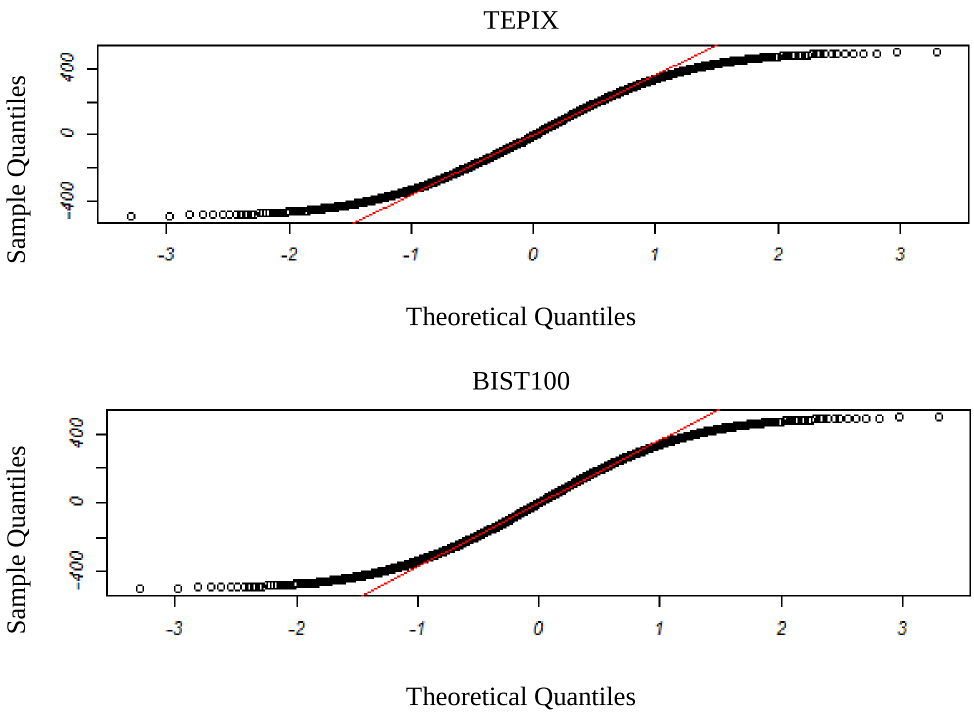


Figure 1: Quantile-Quantile Plot of Daily returns of TEPIX, BIST 100, FGC and USD

3.2 The Marginal Distribution

Since the tested time series return has clustering volatility, for returns empirical distribution compatibility, considering marginal distribution will be necessary. Thus each time series marginal distribution investigated by GARCH (1, 1) and GJR (1, 1) models with t-student and normal distributions has been estimated.

Table 2: Parameter Estimates of GARCH & GIR Model

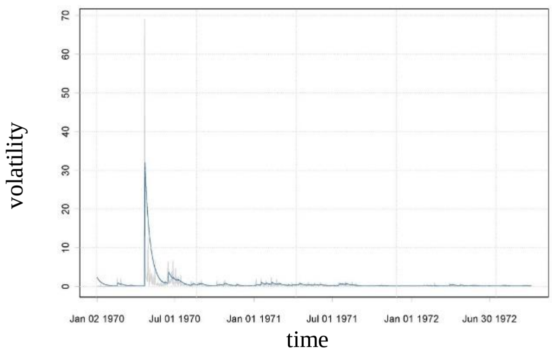
	GARCH				GIR			
	Normal		t-student		Normal		t-student	
Parameter	TEPIX	BIST 100	TEPIX	BIST 100	TEPIX	BIST 100	TEPIX	BIST 100
Ω	0.003	0.029	0.004	0.011	0.003	0.017	0.004	0.009
	(0.00)	(0.96)	(1.39)	(1.00)	(1.55)	(1.29)	(0.91)	(1.14)
α	0.059***	0.161**	0.057***	0.217**	0.053**	0.016	0.050	0.090**
	(0.01)	(2.34)	(2.70)	(3.19)	(2.06)	(0.25)	(1.66)	(2.03)
β	0.888***	0.821***	0.882***	0.782***	0.860***	0.856***	0.841***	0.800***
	(0.03)	(10.55)	(29.27)	(9.30)	(44.02)	(17.01)	(18.76)	(11.75)
γ					0.118	0.235**	0.165***	0.218**
					(1.51)	(2.48)	(10.95)	(3.04)
shape			3.536***	3.717***			3.479***	3.728***
			(7.97)	(10.12)			(6.54)	(9.63)
Log-Likelihood	-452.07	-1125.67	-185.64	-903.68	-444.16	-1111.53	-141.10	-896.11
Ljung-Box test								
Q-Statistics Lag[1]	0.0004	0.7289	0.0003	0.1154	0.0002	0.6771	0.0001	0.0954
P-value	0.98	0.39	0.98	0.73	0.98	0.41	0.98	0.75
Lag[5]	0.0044	4.7432	0.0043	3.6791	0.0031	3.8323	0.0028	2.7241
P-value	1.00	0.44	1.00	0.59	1.00	0.57	1.00	0.74
ARCH LM tests								
Lag[1]	0.0021	0.0752	0.0021	0.0923	0.0020	0.1198	0.0020	0.1274
P-value	0.99	0.96	0.99	0.95	0.99	0.94	0.99	0.93
Lag[5]	0.0052	0.2993	0.0052	0.2200	0.0051	0.3468	0.0051	0.2950
P-value	1.00	0.99	1.00	0.99	1.00	0.99	1.00	0.99
Lag[10]	0.0104	0.5600	0.0104	0.4750	0.0104	0.6435	0.0104	0.6007
P-value	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Notes: The values in parenthesis are the t-Student. Ω is constant of conditional variance equation. α and $\beta$are the parameter of the volatility during the previous period and the variance during the previous period.γ is the GJR's threshold value. Shape is the t-student distribution's degrees of freedom.

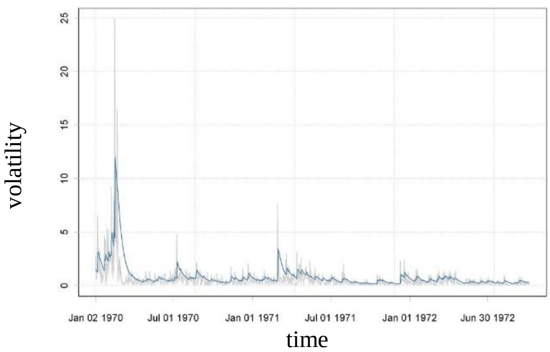
** Significance at the 5% level

*** Significance at the 1% level

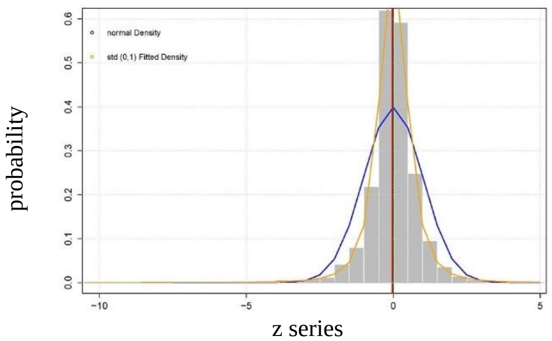
Table 2 reports the results of estimating parameters of the GARCH (1, 1) and GJR (1, 1) models with t-student and normal distributions. The value of α and β are significant at the 5% significance level (excluding BIST 100 normal GIR model) and the Ljung-Box statistic test results don’t reject the null hypothesis of autocorrelations at the 5% significance level; denoting no autocorrelation between the residuals and the estimated ARCH LM Test confirms the lack of heteroscedasticity at the 5% significance level and guarantees the accurate fitted of these models. So, these models present a desirable estimation. In addition, t-student distribution degrees of freedom (Shape), is significant in t-student GARCH and t-student GIR. The γ parameter’s significant and positive values that is the GIR‘s threshold value, in t-student GIR model, is a confirmation of the leveraged effects existence; and indicates that the negative shocks in comparison to positive ones, will have greater effect in volatility market. The empirical and conditional volatility distribution and News Impact Curve of the estimation can be seen in Figure 2.



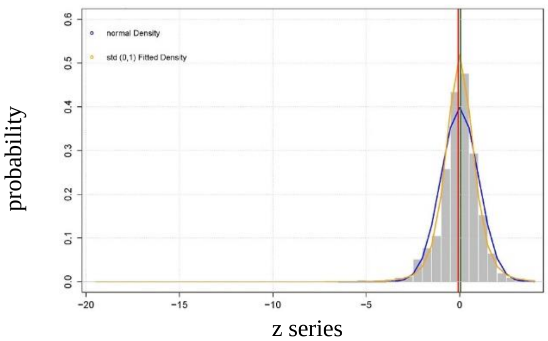
Conditional volatility distribution for TEPIX



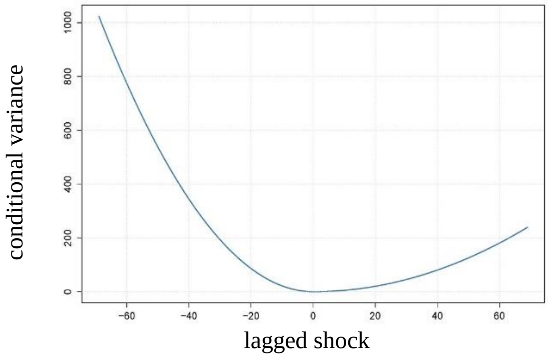
Conditional volatility distribution for BIST 100



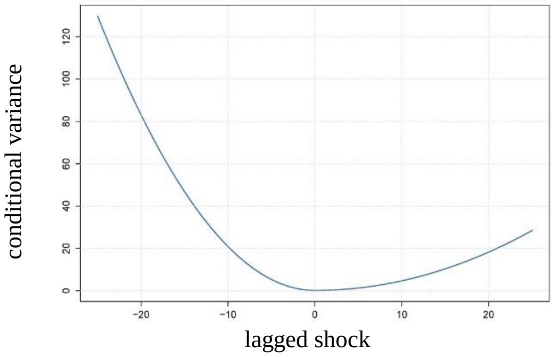
Empirical distribution for TEPIX



Empirical distribution for BIST 100



News Impact Curve for TEPIX



News Impact Curve for BIST 100

Figure 2: News Impact Curve, Empirical &Conditional Volatility Distribution

Thus, according to the log-likelihood parameters value (Table 2), the maximum value of this parameter is for the GIR (1, 1) model with t-student distribution. This result suggests that the GIR (1, 1) model with t distribution, is the best model for TEPIX and BIST 100 time series marginal distribution fit; which can determine the leveraged effects of TEPIX and BIST 100 return in response to the positive and negative shocks. Therefore, in order to calculate the dependence structure between the assets, only the marginal distribution of the process GIR (1, 1) model with t-student distribution is considered.

3.3 The Dependence Structure Modeling Based on the Copula Functions

After estimating the F_i marginal distribution, to determine the data dependence structure, normal copula, t-student copula, Klayton, Gumel and Frank functions for portfolios including TEPIX and BIST 100 return couple has been estimated. In Figure 3, joint return of the portfolio in a range of [0.1] is illustrated.

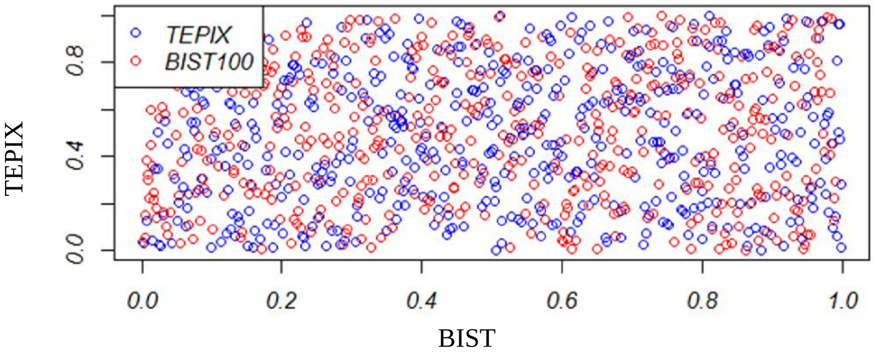


Figure 3: Joint Return of TEPIX and BIST 100

Table 3 reports the estimated parameters of Archimedean copulas, the lower tail dependence coefficient and the upper tail dependence coefficient for index stock market returns.

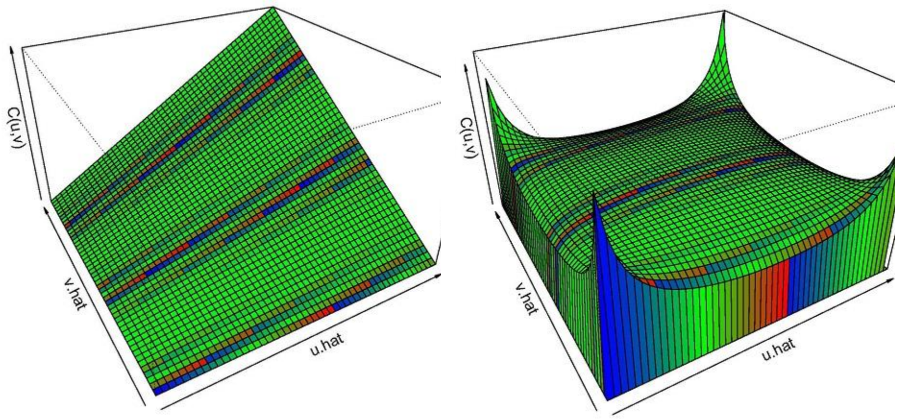
The results of estimating parameters of copula functions, the lower tail dependence coefficient and the upper tail dependence coefficient for index stock market returns are reported in Table 3. The t-student copula function with t-student distribution and GIR (1,1) marginal distribution, with the minimum Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) and the most log-likelihood function (LLF), has the best performance among other copula functions. Thus, two TEPIX and BIST 100 in the form of the portfolio have the upper and lower tails dependence. So, in positive and negative returns, the two indexes dependence of the portfolio will increase. In other words, by considering each positive and negative shock, these two couples will have more dependence.

Table 4: Parameter Estimates for Families of Copula and Model Selection Statistic

Portfolio of TEPIX & BIST 100					
Copula	Normal	t-Student*	Clayton	Gumbel	Frank
ρ	0.0274	0.0299			
θ			0.0299	1.0337	0.165
λ_U	0	0.0440	0	0.0447	0
λ_L	0	0.0440	4.9e-06	0	0
LLF	0.2989	4.5980	0.3391	1.1060	1.1283
AIC	1.4021	-5.1961	-0.2566	-0.2121	1.3217
BIC	6.3168	4.6333	4.6580	4.7025	6.2365

Notes: ρ is the t-student and normal copula's dependence coefficient. θ is the Archimedean Copulas' dependence coefficient. λ_U and λ_L are denote the distribution upper and lower tails dependence respectively. The LLF is the log- likelihood function. AIC and BIC are the Akaike and Bayesian information criterion respectively.

The portfolios probability density (PDF) and cumulative distribution function (CDF), of t-student copula are plotted in Figure 4. The estimated parameters of t-student copula is $\rho = 0.0299$. We observe that the CDF of copula is strongly tailed, which means that the mass for t-student copula is concentrated in this tail and the upper and lower tail have peaks. The estimated coefficient of the upper and lower tails dependence for the pair of Index of stock markets returns ($\lambda_U = \lambda_l = 0.044$) is positive and equal, and indicate that there is no difference in the dependence between TEPIX and BIST 100 returns during bull markets and bear markets. Furthermore, dependence between financial returns is symmetric. This finding is in contrast to the results of Costinot et al. (2000), Patton (2006), Aloui et al. (2011) and Boubaker and Sghaier (2013) who based on the Gumbel copula argue that the dependence structure during boom is stronger than the dependence during recession.



CDF With t-Student Copula PDF With t-Student Copula

Figure 4: CDF & PDF of t-student Copula Estimated for Portfolio

3.4 Conditional Value at Risk Estimation

With regard to determining the appropriate copula function for measuring dependence between TEPIX and BIST 100 return, the estimation of value of the Conditional Value at Risk (CVaR) in the equation 23, will be done easily. For getting the portfolios optimal CVaR amount, first for achieving the optimal weight by using the 5000-day simulation of the portfolio, the optimal weight will be calculated and the CVaR concluded in copula approach (Figure 6).

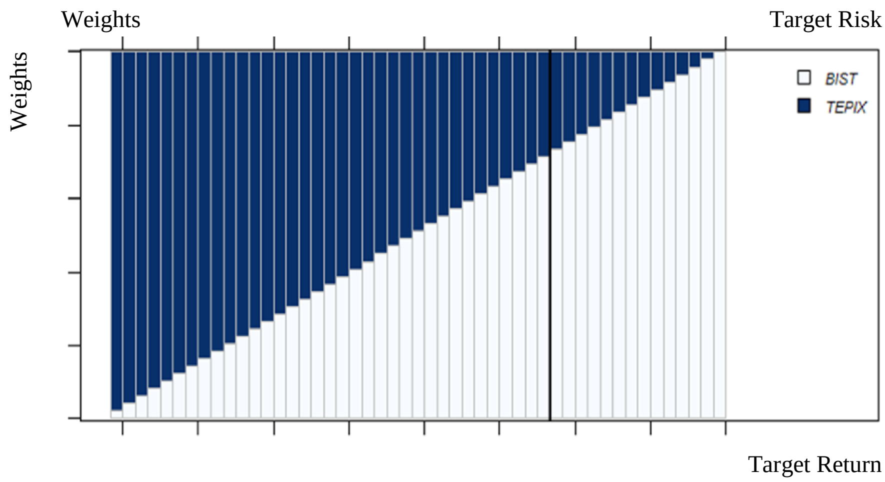


Figure 6: Weights of Optimal Portfolios Based on 5000 Days Simulation

The results of 50 different weights with Variance-Covariance method and concluding t-student copula function dependence for curve of Efficient Frontier of the portfolios can be observed Table 5 for several selective weights.

Table 5: Several Selective Weights of Results of 50 Different Weights

Portfolio of TEPIX & BIST 100						
No. Portfolio	$W_{BIST100}$	W_{TEPIX}	Mean	Cov	VaR	CVaR
1	0%	100%	0.0012	2.0886	5.6069	10.1126
13	26%	74%	0.0009	1.6384	4.3227	7.8633
25	50%	50%	0.0006	1.2964	3.3365	5.8122
37	74%	26%	0.0002	1.1623	3.0438	4.6512
50	100%	0%	0.0001	1.3245	3.5590	5.4564

The more details of the 50 weights, applied in the efficiency curve can be seen in Figure 7. Figure 7 shows that CVaR-Copula is always larger than VaR-Copula when the expected returns are the same. This indicates that it is better to apply CVaR-Copula to capture the dependence structure among assets, so that to estimate CVaR of portfolio more accurately, but VaR-Copula underestimates the risk of portfolio. This evidence is in line with the results in Boubaker and Sghaier (2007) and He and Gong (2009), supporting the finding that CVaR-Copula have better performance from other optimization program. As well as, the optimal weight for a portfolio with the least CVaR, including TEPIX and BIST 100 using the CVaR method and the dependence structure on the t-student copula functions in 37th weight will equal the BIST 100, 0.74 and TEPIX 0.26 weight.

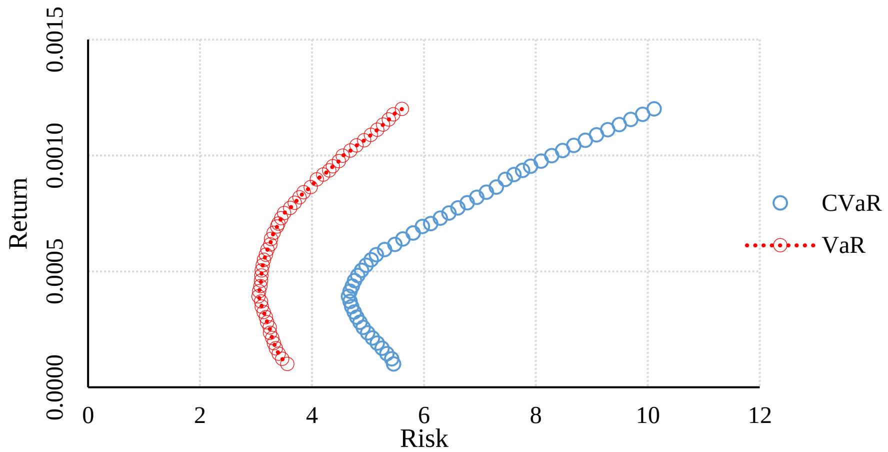


Figure 7: Efficient Frontier of Portfolios of TEPIX & BIST 100 Based on CVaR

4. Conclusion

In this paper, the effect of dependence structure on the optimal portfolio, consisting of TEPIX and BIST 100, has been investigated by the GJR-Copula-CVar model. According to the great importance of the linear dependence structure between the financial assets and their impact on the portfolio, the copula functions approach was used. Since the copula can identify and measure the tail behavior, it‘s a powerful and flexible tool to determine the structure of dependence between highly volatile financial markets and it‘s an alternative for correlation in the financial risk modeling. Therefore, its impact on risk identification and portfolio optimization has been examined in this study.

The results of the empirical studies indicate that t-student copula function is the most optimal function for the dependence structure recognition and two portfolios interpretation. The results of portfolio optimization showed that VaR with Student Copula underestimates the risk of portfolio. In conclusion, the GJR-Copula-based CVaR model can be more accurate than VaR-Copula model.

References

Ang, A., & Bekaert, G. (2002). International Asset Allocation with Regime Shifts. *Review of Financial Studies, 15*(4), 1137-1187.

Ang, A., & Chen, J. (2002). Asymmetric Correlations of Equity Portfolios. *Journal of Financial Economics, 63*(3), 443-494.

Beine, M. (2004). Conditional Covariance's and Direct Central Bank Interventions in the Foreign Exchange Markets. *Journal of Banking & Finance, 28*(6), 1385-1411.

Boubaker, H., & Sghaier, N. (2013). Portfolio Optimization In The Presence Of Dependent Financial Returns with Long Memory: A Copula Based Approach. *Journal of Banking & Finance, 37*(2), 361-377.

Brooks, C., Burke, S., Heravi, S., & Persand, G. (2005). Autoregressive Conditional Kurtosis. *Journal of Financial Econometrics, 3*(3), 399-421.

Chen, Y. H., & Tu, A. (2013). Estimating Hedged Portfolio Value-at-Risk Using The Conditional Copula: An Illustration Of Model Risk. *International Review of Economics & Finance, 27*(C), 514-528.

Das, S., & Uppal, R. (2004). Systemic Risk and International Portfolio Choice. *Journal of Finance, 59*(6), 2809-2834.

Embrechts, P., McNeil, A., & Straumann, D. (2002). Correlation and Dependence in Risk Management: Properties and Pitfalls. In M. A. H. Dempster (Ed.), *Risk Management: Value at Risk and Beyond* (176-223). Cambridge: Cambridge University Press.

Engle, R. (1982). Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation. *Econometrica, 50*(4), 987-1007.

Fantazzini, D. (2008). Dynamic Copula Modelling for Value at Risk. *Frontiers in Finance and Economics, 5*(2), 72-108.

Frank, M. (1979). On The Simultaneous Associativity of F(x, y) and x+y-F(x,y). *Aequationes Mathematicae, 19*(1), 194-226.

Glosten, L., Jagannathan, R., & Runkle, D. (1993). On The Relation between the Expected Value and the Volatility of the Nominal Excess Return on Stocks. *The Journal of Finance, 48*(5), 1779-1801.

Gumbel, E. (1960). Bivariate Exponential Distributions. *Journal of American Statistical Association, 55*(292), 698-707.

Hartmann, P., Straetmans, S., & DeVries, C. (2004). Asset Market Linkages in Crisis Periods. *The Review of Economics and Statistics, 86*(1), 313-326.

Harvey, C., & Siddique, A. (1999). Autoregressive Conditional Skewness. *Journal of Financial and Quantitative Analysis, 34*(4), 465-487.

He, X., & Gong, P. (2009). Measuring the Coupled Risks: A Copula-Based CVaR Model. *Journal of Computational and Applied Mathematics, 223*(2), 1066-1080.

Huang, J. J., Lee, K. J., Liang, H., & Lin, W. F. (2009). Estimating Value at Risk of Portfolio by Conditional Copula-GARCH Method. *Insurance: Mathematics and Economics, 45*(3), 315-324.

Mesfioui, M., & Quessy, J. F. (2005). Bounds on the Value-at-Risk for the Sum of Possibly Dependent Risks. *Insurance: Mathematics and Economics, 37*(1), 135-151.

Palaro, H., & Hotta, L. (2006). Using Conditional Copula to Estimate Value at Risk. *Journal of Data Science, 4*(1), 93-115.

Patton, A. (2006). Modelling Asymmetric Exchange Rate Dependence. *International Economic Review, 47*(2), 527-556.

---------- (2002). *Applications of Copula Theory in Financial Econometrics* (Unpublished Doctoral Dissertation), University of California, USA.

Poon, S. H., Rockinger, M., & Tawn, J. (2004). Modelling Extreme-Value Dependence in International Stock Markets. *Statistica Sinica, 13*(4), 929-953.

Sadique, S., & Silvapulle, P. (2001). Long-term Memory in Stock Market Returns: International Evidence. *International Journal of Finance & Economics, 6*(1), 59-67.

Schmidt, R. (2003). *Dependencies of Extreme Events in Finance: Modelling, Statistics and Data Analysis* (Unpublished Doctoral Dissertation), Ulm University, Germany.

Sklar, A. (1959). *Fonctions de Répartition À N Dimensions Et Leurs Marges*. Paris: Université Paris.

Song, P. K. (2000). Multivariate Dispersion Models Generated From Gaussian Copula. *Scandinavian Journal of Statistics, 27*(2), 305-320.