Iranian Economic Review 2022, 26(2): 369-388
DOI: 10.22059/ier.2020.78380

RESEARCH PAPER

Price Jump Diffusion in Iranian Housing Market (Merton Model and NGARCH Approach)

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Received: 29 June 2020, Revised: 18 August 2020, Accepted: 12 September 2020

Abstract

This paper aims to model the housing price behavior in the Iranian market using three stochastic differential equations: the Black-Scholes model, Merton model, and geometric Brownian motion with nonlinear GARCH. The data of this study include monthly observations on housing prices between 2009 and 2018. Also, to estimate the coefficients of equations, we used the maximum likelihood approach, and the drift and diffusion parameters were calculated. The findings suggest that the efficient-market hypothesis does not hold in the Iranian housing market since the sudden jump under systematic risks is indicative of an increase in inefficiencies in the housing market. In this paper, we also use the nonlinear GARCH (NGARCH) model based on the Merton model to investigate the impact of good and bad news and positive and negative shocks. According to the results of the NGARCH model, housing price is more affected by bad news and negative shocks. According to the estimated equations in the Iranian housing market and the maximum likelihood function, the geometric Brownian model with stochastic NGARCH-based fluctuations has more explanatory power than the Merton model and the geometric Brownian model with constant volatility.

Keywords: Housing Price, Merton Model, Black-Scholes Model, Nonlinear GARCH Model, Jump-Diffusion Model.
JEL Classification: C32, E31, E58.

Introduction

Housing is a type of asset that has severe measurement problems in its valuation, caused by heterogeneity on both structural and locational attributes(Cho, 1996). Almost in all economies across the world, housing plays a crucial role and is considered to be a significant part of households' wealth. Modeling the behavior of dwelling prices is essential for economic policymakers, households, and market players, while price fluctuations and bubble formation in the housing market can as severely impact the GDP and other macroeconomic variables as the economy of families and social welfare. Unlike the other assets, searching for the desired dwelling according to one's budget and utility is not a simple matter. However, when many frictions associated with the purchase of housing, it is hardly surprising that observed behavior deviates from that predicted by simple models of the economic market(Hwang and Quigley, 2010). Researchers have tried to exploit various mathematical and quantitative approaches to explain the underlying patterns of house pricing. In (Meen, 2002), the author argued that using a common methodology on time-series housing market models can describe the behavior of house price elasticities in the U.S.A and the U.K so that the similarities are more striking than the differences. An extensive examination of early researches on the effi-

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cient market hypothesis with time-series models has been conducted in (Cho, 1996). In order to address the pricing of the fixed-rate mortgage with prepayment and default options, authors in (Calvo-Garrido and Vázquez, 2015) assume that the house price dynamics is governed by jump-diffusion models of Merton (R.C. Merton, 1976; Kou, 2002), while the presence of a finite number of jumps following a Poisson process is the other consideration of the research.

The studies have significantly been expanded since the Black-Scholes(BS) model has become the most favorite technique for option pricing (Scholes and Black, 1973). Although the BS model is still considered as a popular classical method to model the market, empirical finding shows that the model lacks the capability of capturing some essential characteristic features of prices, e.g., long-range correlations, periods of constant values, heavy-tailed and skewed marginal distributions and so on. A generalized BS model whose price of an underlying asset follows a time-changed geometric fractional Brownian motion derived in (Gu et al., 2012) in order to display the long-range dependence observed in data points. Therefore, the authors obtain the option formula for the European call option when the timestep is given. In an incomplete market structure and assuming no-arbitrage pricing principle, the pricing of European, American and Butterfly options under two-state and three-state regime switching Merton jump diffusion model (Merton, 1973) has been explored in (Rambeerich and Pantelous, 2016) based on a system of partial integro-differential equations (PIDEs) using Lagrange finite element and exponential time integration methods (Rambeerich et al., 2009) and (Rambeerich et al., 2013). The existing literature on Markovian regime-switching models follows a common assumption that the Markov chain is independent of the Brownian motion. Authors in (Dang et al., 2016) argued that this assumption does not comply with realistic situations, e.g., for options written on major indices. Thereafter, they proposed a PDE numerical schemes for pricing options under the conditions that the underlying process follows a very general state-dependent regime-switching jump-diffusion model. They concluded that the numerical results indicate that hedging under their model could be very different from a state-dependent model.

The random walk pattern is one of the most important patterns of simulating housing price behavior. Applying the hypothesis that house pricing follows a random walk in (Englund et al., 1999), the authors conclude that the model can be rejected against the alternative assumption of first-order serial correlation. Therefore, the results from any repeat sales model ignoring serial correlation are being considered in doubt. A random walk in the continuous model is shown in the form of the Brownian process. Using stochastic differential equations, researchers have succeeded in simulating the behavior of asset pricing (Brennan and Schwartz, 1985; Gibson and Schwartz, 1990; Heston, 1993; Schwartz, 1997). These equations are referred to as the "diffusion process".

Providing a model for predicting economic markets turbulence, pricing risks, and economic phenomena, which are a clear example of stochastic phenomena, requires a careful analysis that physicists in the field of statistical physics and complex systems have played a significant role in this area. A clear example of this is the simulation of market behavior with Brownian motion, which the mathematical models used to explain it, are the fundamental tools on which all financial asset pricing, options, and derivative pricing models are based (Ermogenous, 2006). It should be noted that the assumption the housing market follows the traditional geometric Brownian motion, is only reasonable under relatively stable prices. Assuming that the agents in a housing economy are risk-neutral, the authors in (Bardhan et al., 2006) proposed a new option-pricing method. Their assumption reduces the problem to comply with the standard Brownian motion. In order to properly model the housing prices process, most of the discrete-time models have been of GARCH type, while the continuous-time models were based on diffusion models (Chen et al., 2010). Estimating parameters of the jump-diffusion model (JDM) using expectation-maximum (EM) gradient algorithm, the authors in (Chen et al.,

2010) investigated the jump parameters of the housing price to represent the sever impacts of the abnormal volatility of jump size on the mortgage insurance premium in the arrival of bad news about the housing market.

Over the course of years of study on the financial markets, the diffusion processes have become an important part of the literature on financial economics. In particular, in the core of financial economics, diffusion processes are used to model the housing price, interest rates, asset pricing, financial derivatives pricing, value at risk, optimal basket selection, and fluctuation modeling. Spatial and temporal dependence of idiosyncratic housing prices were explicitly incorporated into a repeat transaction model to develop a model of price diffusion (Hwang and Quigley, 2010). In most models, stochastic differential equations are used to explain the behavior of economic variables. In the first models presented, the underlying asset has followed the zero-mean Brownian process. The most important feature of the proposed models is the behavior of the underlying asset. In some patterns, the underlying asset has a continuous path over time, and in some other patterns, the underlying asset has a discrete path with a price jump. In these patterns, the impact of the entry of information and news related to the underlying asset in the financial market are taken into attention. Recently a utility indifference-based model by extending Merton’s optimal portfolio model proposed to investigate the pricing issue of house rents under housing market uncertainty in order to provide a better understanding of the relationship between housing prices and rents (Fan et al., 2018). To deal with transaction costs, the authors in (Tian and Zhang, 2020) transform the pricing problem into solving a nonlinear PIDE based on an effective call in time and global time method introduced in (Dewynne et al., 1994). In order to model pricing discrete barrier options in the presence of liquidity risk and jumps, authors in (Li et al., 2019) extended the classical jump-diffusion models in (Merton, 1976) and (Kou, 2002) to include the effects of liquidity costs and jump risk on the underlying asset price. To provide an efficient simulation scheme for Value-at-Risk calculation under a jump-diffusion model, (Fuh et al., 2018) computed a portfolio’s VaR estimating loss probability for assets with jump risks by Monte Carlo simulations. In a recent study, a maximal overlap discrete wavelet transform (MODWT) used to identify the time-variant jumps in financial series, while a reinforcement learning approach deployed for optimal weight identification (Chen et al., 2019).

(Nademi and Khochiany, 2017) studied the relationship between the housing sector and some macroeconomic variables in Iran. The purpose of their study was to investigate the relationship between housing sector indicators and some macroeconomic variables in Iran with the wavelet coherence approach during years 1992-2015. The results of wavelet coherence diagrams indicate the negative correlation between the housing price index and economic growth in the period 2012 to 2015 in the one-year time scale, while during the years 2006 to 2010, the correlation between these two variables was positive and the housing price index was the cause of economic growth. Also, the intensity of the correlation between economic growth and land price index in short-term horizons from 1993 to 1996 were negative and from 2001 to 2004 were positive. Moreover, the inflation rate and the housing price index also have positive correlation and in the short and mid-term, the inflation rate has been the cause of housing price fluctuations. The authors concluded that, the volume of liquidity and price indicators of the housing sector have positive correlation in the long run.

(Abolhasani et al., 2016) Studied the effect of monetary shocks and oil shocks on production and housing inflation in the Iranian economy. their study aims to identify price and production fluctuations in the housing sector and intends to design a New-Keynesian dynamic stochastic general equilibrium model considering the housing sector for the Iranian economy through which to identify the impact of monetary shocks and oil shocks on fluctuations in this sector. The results of the instantaneous reaction functions show that the increase in the money supply growth rate has caused a temporary increase in production and inflation in the housing and non-housing sectors. Meanwhile, due to the higher supply elasticity in the production of non-housing goods and services, the effect of the monetary shock on the production of the non-housing sector is greater than the housing sector. Oil revenues increase inflation in the economy by increasing liquidity and increasing demand from the private sector and households. The results show that the occurrence of an oil shock causes a temporary increase in production and inflation in the housing and non-housing sectors. The difference is that the inflationary impact of this shock is greater than production. The authots claim that the results of the instantaneous reaction functions and the comparison of the model moments with the real data show that the proposed model can largely explain the cyclical fluctuations of macroeconomic variables in the housing and non-housing sector.

(Mousavi and Doroodian, 2016) have analyzed the factors affecting housing prices in Tehran. In their study, the impact of land price, construction cost, real interest rate, per capita completed residential buildings, liquidity, and efficiency of competing markets have been measured using quarterly data. The structural time series method and Kalman filter algorithm are used to estimate the unknown parameters using maximum likelihood method. The results show the significant negative impact of real interest rates, the return on alternative assets (gold, currency, stocks), per capita of completed dwelling units, and the positive impact of construction costs along with the insignificant impact of the liquidity growth.The strong relationship between land and housing prices is more related to the comovement of these two variables than to the causal effect. Among the policy implications of the results is that the control of housing market fluctuations should be focused on strengthening the supply process and controlling construction costs as two major endogenous factors impacting housing prices.

(Gholizade and BakhtiariPour, 2012), while focusing on the price shocks of the housing market in the years 2006-2007, have studied the effect of bank facilities on housing prices in Iran during the period 1991-2007. The results of their estimation using the self-explanatory vector model with wide intervals also confirm the existence of a positive correlation between housing facilities and housing prices in the long and short term. Due to the low elasticity of housing prices in the period of price increase compared with credit facilities in contrast to the other variables, credit growth will not have a strong effect on housing price increases. Therefore, policymakers should not worry too much about increasing credit facilities to the housing sector, even if there is a strong connection, complementary policies can address this concern.

Gholizade and Noroozonejad (2019) examined the relationship between housing prices and business cycles in Iran. Using quarterly data of the period 1991-2016, the authors investigated the relationship between housing prices, investment, and economic fluctuations by applying a dynamic stochastic general equilibrium model. They showed that the correlation between housing prices and commercial investment is influenced by the housing price dynamics at the macroeconomic level.

This paper is primarily concerned with the behavior of the housing price in Iran under the Black-Scholes model, the Merton model, and the non-linear GARCH (NGARCH) model based on the Merton model. To the best of our knowledge, there is not any serious study using the same approach on housing prices in the Iran market. The reason that we applied stochastic approaches to conducting our study is to evaluate whether the efficient-market hypothesis maintains in the Iran housing market or not. Due to the fact that the housing market in Iran, and especially in big cities around the country, has drastically fluctuated during the past decade, housing policy-making in Iran needs to be able to determine an appropriate model encompassing the market. Applying the non-linear GARCH model based on the Merton model brings more evidence if there is any asymmetric behavior in the way that market players play their roles.

The rest of the paper is organized as follows. In Section 2, we provide the reader with a close look at the housing market in Iran, introducing both intrinsic and extrinsic impact factors on the market. For this purpose, we examined the impact of several significant economic factors on the housing market in a twenty-year period from 1997 until 2007. In Section 3, the theoretical foundation of related works on pricing models will be briefly overviewed to illustrate the methodology of our research work. In Section 4, we examine the selected pricing models, geometric Brownian motion, Merton model, and Merton model based on the nonlinear GARCH model using Iran's housing market data to present some numerical results allowing to compare the three models. A final conclusion will be delivered in Section 5.

An Insight into Iran Housing Market

Factors affecting housing prices are put in two major endogenous and exogenous categories.

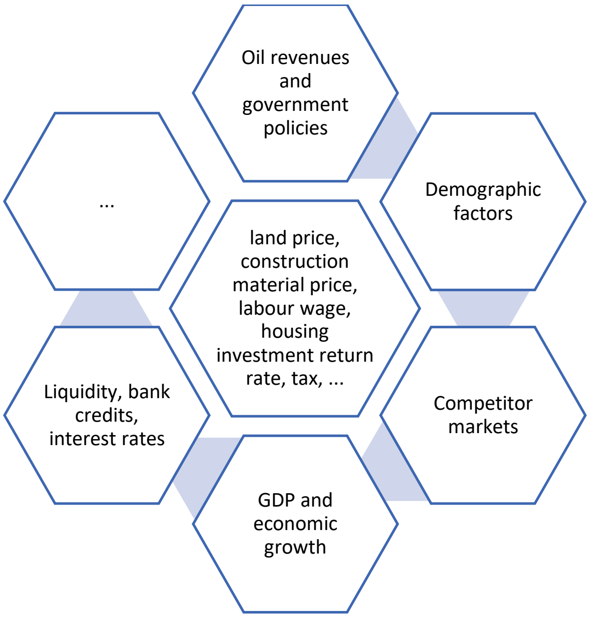

Figure 1. Endogenous and Exogenous Factors Affecting Housing Prices
Source: Research finding.

Endogenous factors mainly impact the total cost of the dwelling units. The price of habitable land plays an important role in determining construction costs. In large cities, the lack of suitable land is the most important constraint on the construction and supply of new housing units. The price of construction materials and labor wages are the second most significant endogenous factors in the production of housing and constitute a major part of construction costs. The price of construction materials is one of the channels that the exchange rate and the price of energy impact on the housing market. An increase in the exchange rate and the price of energy increases the cost of production and transportation of construction materials causes the increase in construction costs in the housing sector.

On the other hand, exogenous factors cover more dimensions including macroeconomic variables, e.g. oil revenues, interest rates, demographic factors, and so on. In developed economies, monetary policy is the major external factor affecting housing price fluctuations, while in oil-exporting countries, oil-price changes are recognized as the most significant one. Therefore, as the share of the oil revenues in the government budget and the country's economy increases, the transmission channels of oil price shocks on the domestic economy will be more strong. Another external factor affecting housing prices is bank deposits, and as a result, the central bank's monetary and credit policies play a crucial role in housing demand. Specifically, if the liquidity increases while the interest rates decrease, the housing demand and consequently its relative price will experience higher figures. Besides, housing facilities, whether purchase or construction programs, directly stimulate the supply and demand of this sector. The growth of the monetary base and the money multiplier are the two significant influential factors on domestic liquidity growth. In recent years, the banking system is the major source of the increasing liquidity ratio across the country, while the debts of the commercial banks to the central bank resulted in the growth of the monetary base. Demographic changes play an essential role in determining housing demand, especially in the long run. The impact of these developments is often on the consumer housing demand, and the capital demand is less affected. Therefore, short-term fluctuations in the housing market have little to do with these developments. The environment issue, especially the water crisis, has not affected the supply and demand of housing units, at least in cities, but in the long run, it can become a critical factor in the habitability of different geographical areas. However, the main objective of the investment is to maintain the value of assets by investing in different markets to earn the highest return by choosing the best portfolio. In developed countries, the diversity and efficiency of the capital market and its derivatives cause the existing capital in an organized capital market to be attracted and directed towards the financing of enterprises. The lack of a developed capital market in Iran has caused a large part of the country's capital to flow into the housing market and there is no other market in the dimensions of this market. A major competitor for the housing market is known to be the bank deposits, therefore, any recession in the housing market results in increasing deposits rather than leading to more investment in stock or other capital markets.

According to what mentioned in the above paragraphs, the housing sector in Iran plays a vital role in the country's economy and weigh much more for people and government compared with many other countries. In Iran, due to the underdevelopment and inefficiency of financial and capital markets, the housing market has played a significant role in attracting huge investments. Therefore, it is not unrealistic to say that the housing market is representative of the asset market, which has substantial forward and backward linkages with other economic sectors. According to a recent estimation, residential and commercial properties make up an average of more than 80 percent of Iran's net worth. The housing portion in the Iranian household cost basket has significantly increased, from 28 percent in 1992 to more than 33 percent in 2013, while the share is higher in the expenditure basket of lower decile households than in upper deciles and in cities more than in rural areas. Housing also plays a crucial role in calculating the CPI consumer price index, and its coefficient of importance equals to 30.14 percent, according to the base year of 2011, including 7.51 percent for residential rental houses, 20.92 percent for the rental value of a private residence and 1.71 percent for building services. One of the most significant factors influencing the housing market is monetary policies. Whenever there is a big gap between the growth rate of liquidity, and the growth rate of housing prices, the fluctuations in the second, typically follow the first. In the short term, any increase in liquidity results in the higher demands in the housing market, which in turn results in the relative prices' growing. However, with 4 to 8 seasons of interruption, investment in the market goes up, resulting in an increase in supply, which will moderate the relative prices. However, the impact of monetary policies on housing prices in Iran is significantly lower compared with the impact of oil shocks. In the rest of this section, we demonstrate how the fluctuations in significant macroeconomic variables such as liquidity, inflation, foreign exchange rate, and some others result in changes in housing market prices. We extract the data needed for this section from Economic Research and Policy Department of Central Bank of Iran[1].

1. https://tsd.cbi.ir/DisplayEn/Content.aspx

Figure 2. Percentage of Growth Rate Changes in Liquidity and Housing Price
Source: Iran's Ministry of Economic Affairs and Finance.

Figure 2 shows fluctuations in the growth rate of liquidity and the average growth rate of housing prices. Maintaining asset value by investing in different markets and earning the highest returns by choosing the best asset portfolio is the primary goal of the investor. In Iran, the view of most people is that housing is one of the best investment options and has less risk than others. Therefore, people often prefer to purchase a private house as a kind of safe investment. In developed countries, however, the diversity and efficiency of capital markets and their derivatives make the capital available for production and economic growth. Therefore, in developing countries such as Iran, instead of realtor activities, they should focus on goals such as financing industrial and SME enterprises. The lack of a diversified and developed capital market in Iran has led to more capital inflows into the housing market, and there is virtually no other substituting market that can absorb the shocks to the economy to reduce the impacts on the other sectors. There is a long list of factors impacting the real estate market, some have been classified as internal, and the others influence the market as external factors. The First category affects the volume and the way demand and supply of the market changes impacting the final construction price. Extrinsic factors, however, are derived from outside of the market and mainly impact housing demand and prices through influencing market attractiveness and consumer power purchase.

Land price is one of the most critical factors influencing housing prices in Iran; However, land supply limitations because of inhabitable land shortcoming, and keeping them in order to achieve higher capital gains, play a critical role in housing price changes. On the other hand, the lack of speculation tax for properties in the country made this market more attractive for speculators.Figure 3 shows a significant positive correlation between the average price growth rate per square meter of land in old, dilapidated houses and the average price growth per square meter of residential units in the selected cities of the country.

The trend in Figure 3 provides essential insights into the volatility of these variables during booms and busts. As can be seen in this graph, the price growth cycles of both variables are in harmony with each other, but the change in the price of land in old buildings fluctuates more than the change in the price of residential units. In other words, during the boom, the increase in the price of land has increased more than the rise in the price of residential units, and during the recession, the price of land has decreased more than the price of residential units. The moving average trend line of the two variables clearly indicates this.

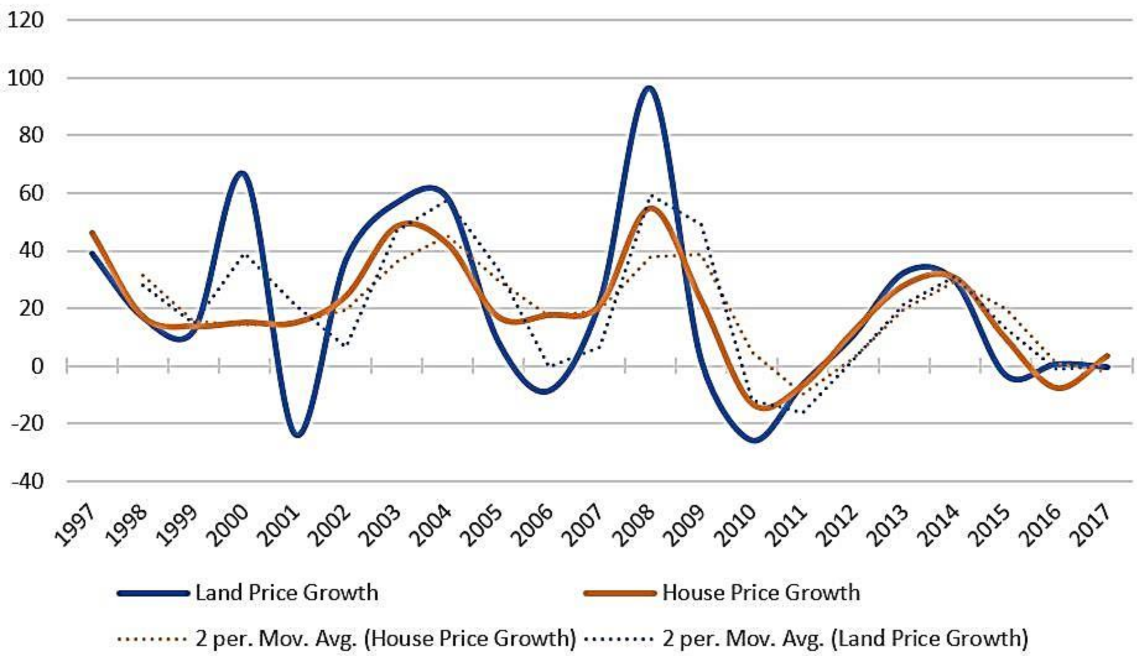

Figure 3. Average Growth Rate of Real Estate Prices from 1994 to 2016
Source: Research finding.

This is mainly due to the increasing tendency of investors to build during the boom and the decrease in their willingness to invest in housing during the recession. Thus, during the boom period, the purchase of land goes significantly up, and the expectation of an increase in its price will lead to a further jump in the price of land. While during the recession, consumers are more likely to buy residential units instead of construction, which in turn reduces demand for land for construction. It further adjusts its price and, on the other hand, prevents housing prices from falling sharply for consumer housing units through an increase in residential houses demand. **Figure 4** shows the relationship between two variables, investment growth in new buildings in urban areas at the early stages of construction, and the land price growth. The correlation coefficient between the two variables, i.e. 0.7675, occurred in the same period, which, while expressing a strong positive correlation between them, confirms the above claim that land prices fluctuate during the boom and stagnation of investment in the housing market, the fact that has been proved by a comparison between land price fluctuations and new investment in construction.

Figure 4. New Investment in Construction vs. Land Price Growth Rate; A One Year Lagged Spearmen Rank between Two Variables Equals 0.4526 with a p-value of about 0.0466 Means Strong Positive Correlation

Source: Research finding.

Another intrinsic factor in housing price fluctuations is construction costs. According to the information on the period of this study, the index of construction services, which is related to the wages of workers, has had a direct relationship with the price of housing. During this period, the correlation coefficient between the growth of the construction services index and the growth of housing prices was 0.4377, with the p-value of 0.048, which indicates the direct correlation between the two variables. The critical point in this regard is the difference between the standard deviation of housing price growth and the growth of the construction services index, which is equal to 18.06 and 7.5, respectively. This indicates fewer fluctuations in the cost of construction services, which seems natural, and confirms that changes in housing prices are influenced by factors other than construction costs. In other words, the factors affecting housing prices do not necessarily lead to an increase in housing prices through the rise in construction costs. Besides, contrary to the growth of housing prices, the growth of construction services at any point in the period under review has not been negative, which in economic literature indicates the adhesion of nominal wages to the bottom (Figure 5). Another essential component of construction cost is the cost of building materials. The correlation coefficient between the annual growth of the price of selected building materials and the growth of the price of housing is equal to 0.4779.

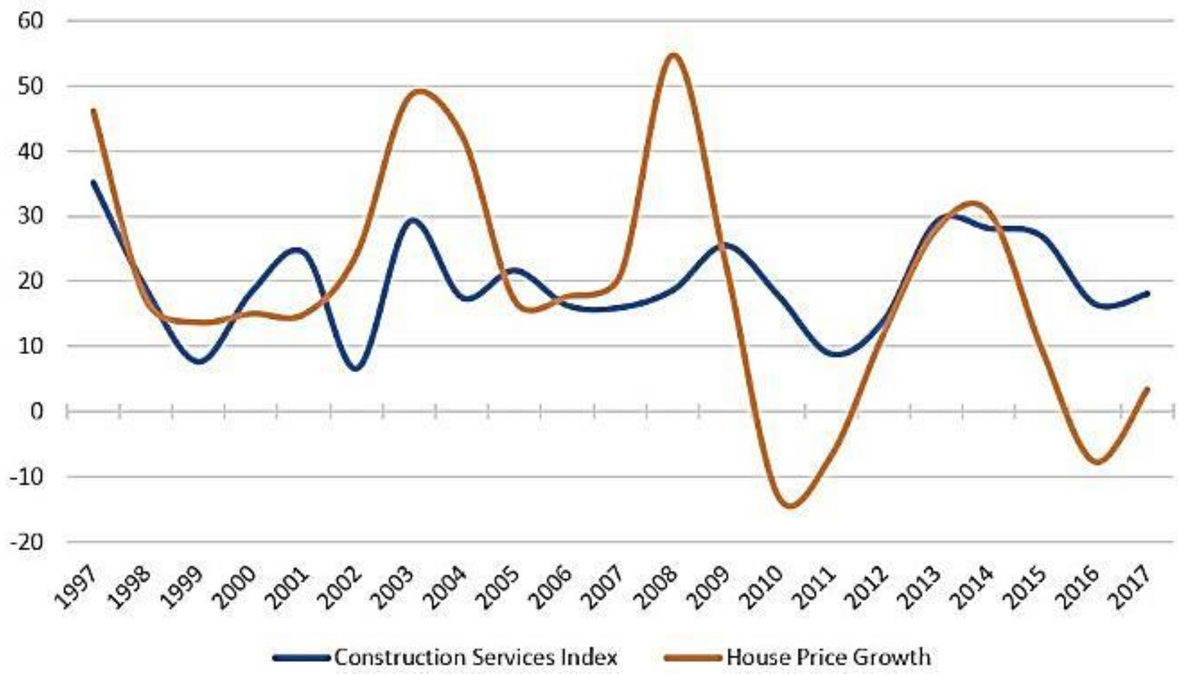

Figure 5. House Price Growth Rate and Construction Services Growth
Source: Research finding.

As can be seen in

Figure *6*, in the years when the cost of building materials has been rising, the price of housing has also experienced a proportionate growth. For example, in 2008, when the price of construction materials increased by about 47%, the price of housing also increased by more than 54%. However, similar to the argument made about the construction price index, despite the strong correlation between the two variables, the reason for the increase in housing prices in the country cannot be attributed mainly to the rise in the price of construction materials and construction costs. On the other hand, increasing the demand for housing, which is mainly associated with the growth of its price, can also increase its price by creating pressure on building materials.

The impact of interest rates on demand and investment in the housing sector is not high. In general, increasing the supply of money, mainly by growing profitable activities and investing in the housing market, increases housing prices volatility. In recent years, the central bank's interest rate policy has been mostly ineffective in regulating interest rates in accordance with a common monetary policy. Therefore, it has provided an incentive for liquidity to spill-over into the housing sector, given the higher inflation rates, especially in the housing sector, compared with the interest rate on loans and interest on bank deposits. Therefore, in Iran, instead of the housing sector be highly sensitive to monetary policy, it mainly signals a change in the price and income of capital related to this sector. Due to the higher capital gains of housing compared with interest rates and depreciation rates, people are more likely to purchase houses obtaining bank loans despite high interest-rates. Therefore, due to the weakness of capital and bond markets in the country, many financing needs are met through the banking sector. Thus, the expansion of credit in this sector, as well as the instability of the country's foreign exchange reserves, which are mainly dependent on oil revenues, is the most crucial factor in the growth of the money supply.

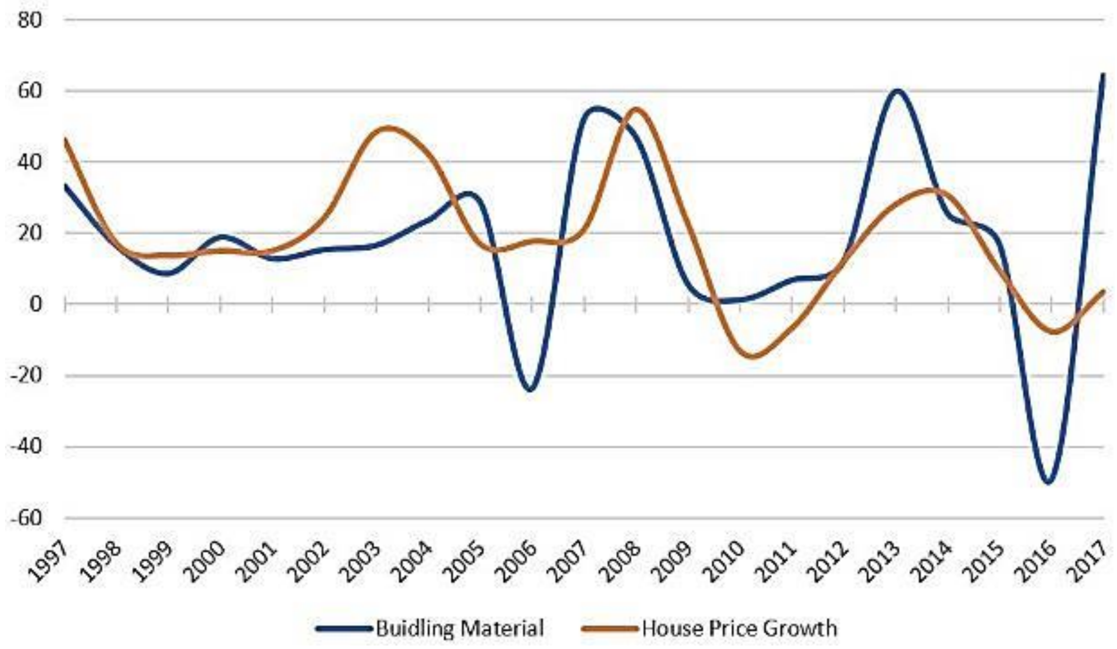

Figure 6. House Price Growth Rate vs. Percentage Changes in the Price of Building Material
Source: Research finding.

Meanwhile, the impact of rising oil revenues on the money supply and the fluctuations in housing prices in the country is significant. In general, the behavior of the housing market in oil-dependent countries is different from other countries, so that oil shocks in these countries are one of the leading causes of housing price fluctuations. In these countries, the level of liquidity in the society increases mainly due to the lack of proper management of the growth of foreign exchange earnings from crude oil exports and the overflow of these revenues to the economy by converting them into the common currency of the country. In the years when the country's oil revenues have been growing, liquidity has also experienced growth with a time lag. For example, in 2000, the growth of oil revenues was equal to 96.1 percent, the effect of which on the growth of liquidity is seen since 2001.

According to Figure 7**,** Another period of significant growth in the increase in oil revenues is 2011 through 2012; thereafter, liquidity has experienced growth in 2012 and 2013. In general, this increase in liquidity is mainly accompanied by pressure on the demand of the whole economy, part of which is met by imports. However, since housing is one of the non-tradable goods, it is not possible to import it, and on the other hand, it is not possible to increase its production and supply in the short term. Therefore, the lack of response to the demand for housing in the short term is one of the factors that increase its price. Besides, the increase in imports of tradable consumer goods mainly results in weakening of the country's production capacity in these goods, which also makes it unattractive to invest in manufacturing sectors and thus overflows capital to the real estate and housing sector - as a substitute. The influx of liquidity and capital into the housing market incorporates a two-fold impact on it: buying existing housing and increasing investment in new buildings, which has an opposite long-term effect on the dwelling market. Firstly, in the short term, the increase in housing demand due to the existence of wandering liquidity results in an upsurge in housing prices. This makes housing attractive and intensifies its speculative demand as a capital good, which has a growing impact on housing prices. Secondly, this growth in housing prices increases the attractiveness of investing in housing construction and thus increases supply.

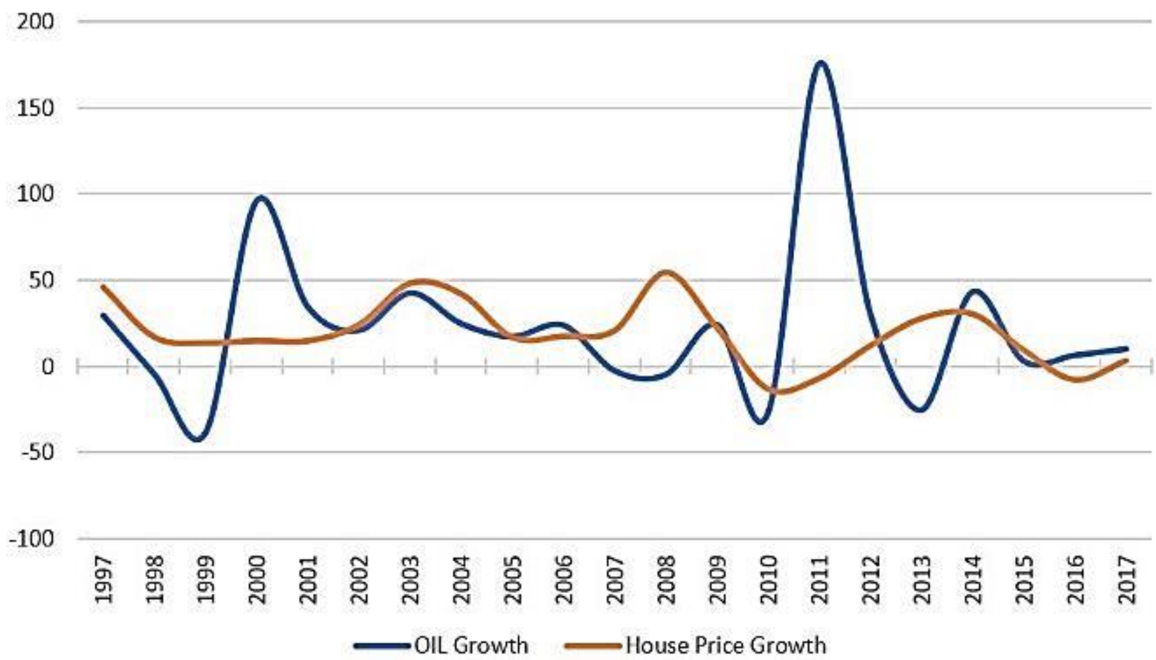

Figure 7. Oil Income Impact on Changes in Housing Market
Source: Research finding.

This is the main factor introducing short-term imbalances and creating cycles of booms and busts in the country's housing market. As it can be seen in Table 1, the value of the time-lagged correlation coefficient between liquidity and housing price variables is equal to 0.4647, which shows a direct and moderate correlation between them with a one-year delay. Accordingly, increasing liquidity with a one-year break has an increasing effect on housing prices. In addition to the common effect of increasing oil revenues on the volume of liquidity in the Iranian economy, the opening of credit lines for the implementation of the Mehr housing project in the same period was accompanied by an increase in the monetary base, which is mainly based on abundant oil revenues.

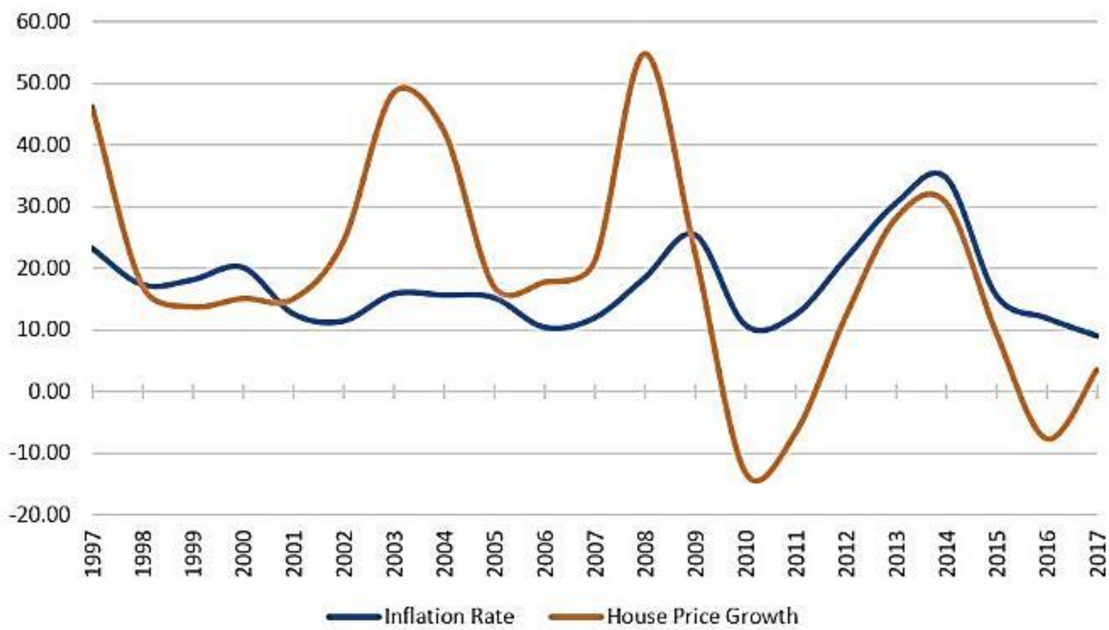

Figure 8. Co-movement between Inflation Rate and Housing Price
Source: Research finding.

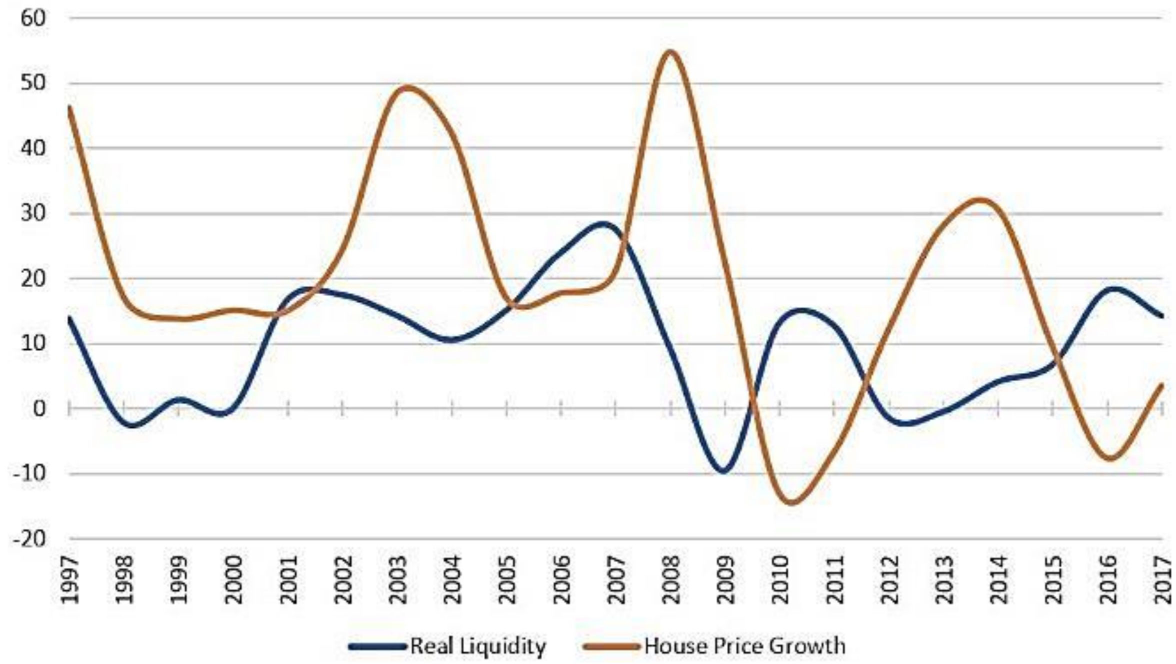

Figure 9. Real Liquidity and Housing Price Has a Significant Positive Correlation
Source: Research finding.

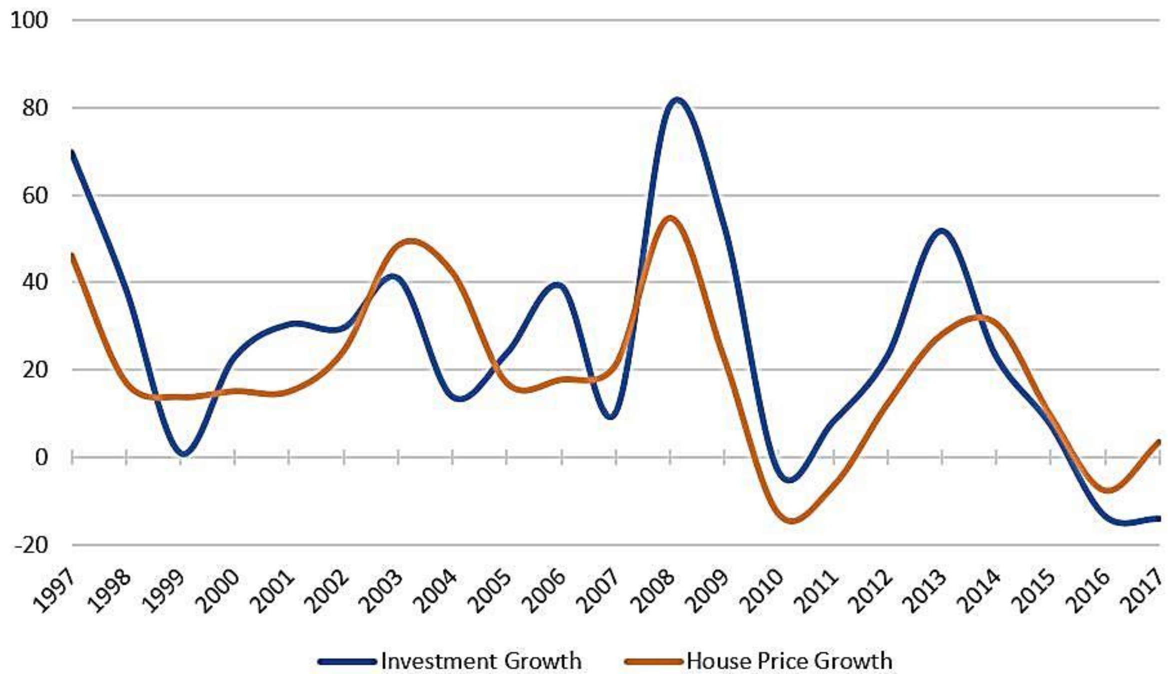

Figure 10. Investment in New Building Construction Increases as Prices in the Housing Market Surge
Source: Research finding.

According to economic theories, one of the factors in increasing liquidity is the growth of the monetary base, which increases liquidity through the banking system functionality and the mechanism of the increasing monetary coefficient. This issue has been recognized as a growing factor in the inflammation of the housing market during the 2010s. Increases in housing prices at the macroeconomic level can be an indicator of market trends and needs, and even as a predictor of macroeconomic variables such as inflation (Goodhart and Hofmann, 2008).

As shown in Figure 8, the inflation rate and the rate of change in housing prices after 2009 are more consistent than in previous years. In addition, the study of the correlation coefficient between these two variables clearly confirms this issue so that before 2009 the correlation coefficient was about 0.12 (p-value=0.72), while in the period after 2009, it reached 0.9 with the p-value equals to 0.002. Therefore, it can be said that the increase in liquidity during the period after 2009 has mainly absorbed in the housing market and stimulated its price so that its spillover effect on other economic sectors has also played a significant role in increasing the general inflation rate of society.

Figure **9** shows the two-year lagged co-movement of housing prices with real liquidity. The correlation coefficient index also indicates a moderate relationship between these two variables with two years of real liquidity priority. According to Table 1, Spearman rank of these two variables equals 0.5017, where the p-value is less than 0.05, i.e., 0.0303. This means that a considerable amount of real liquidity has been invested in the housing market. The second stage can be explained by the effect of increasing oil revenues on the housing market by the relationship between housing prices and investment in the construction sector. As shown in **Error! Reference source not found.**, changes in housing prices are accompanied by changes in private sector investment at all stages of construction.

It can be seen that during the period when housing prices were rising, people's incentives to invest in construction increased, and conversely, in periods when housing prices or price growth were declining, investment grew less or even experienced negative growth.

Another factor influencing housing prices is bank deposits, which are known as the most important competing market for housing in the country. As shown in **Error! Reference source not found.**, low-interest rates on bank deposits relative to inflation, especially during the boom in the housing market, increase the flow of liquidity to this market and thus increase its price, however, since the bank deposits benefit from higher rates on interest compared with the inflation rate, people are more willing to deposit their money, results in intensifying the real estate market recession. Therefore, this factor also plays a significant role in creating cycles of boom and bust in the housing market.

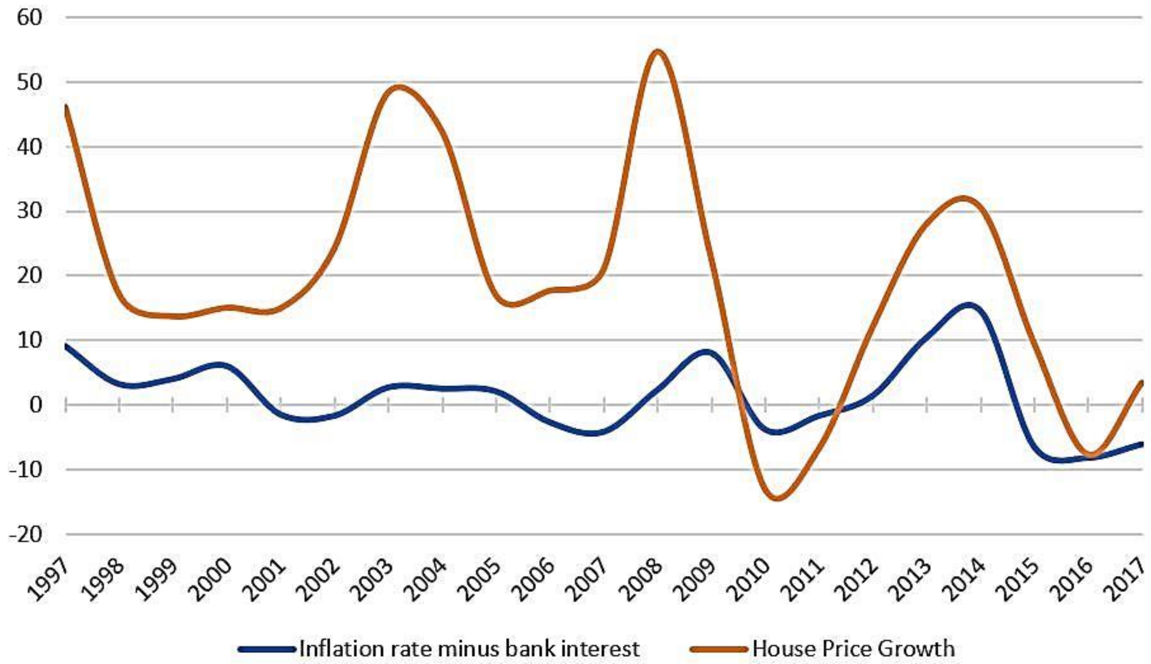

Figure 11. Bank Deposits are Amongst Strong Competitor Markets for the Housing Market
Source: Research finding.

Exchange rate fluctuations are also considered as competitors and alternatives to the housing market, although the effect of the exchange rate on the housing market is less than other factors. The contradictory impact of the exchange rate in two distinct ways moderates its role in changes in housing prices. While the attractiveness of increases in the exchange rate may have the opposite impact on the flow of liquidity to the housing market and thus push housing prices lower, on the other hand, it plays a role as an intrinsic factor in increasing Housing prices by influencing the costs of construction.

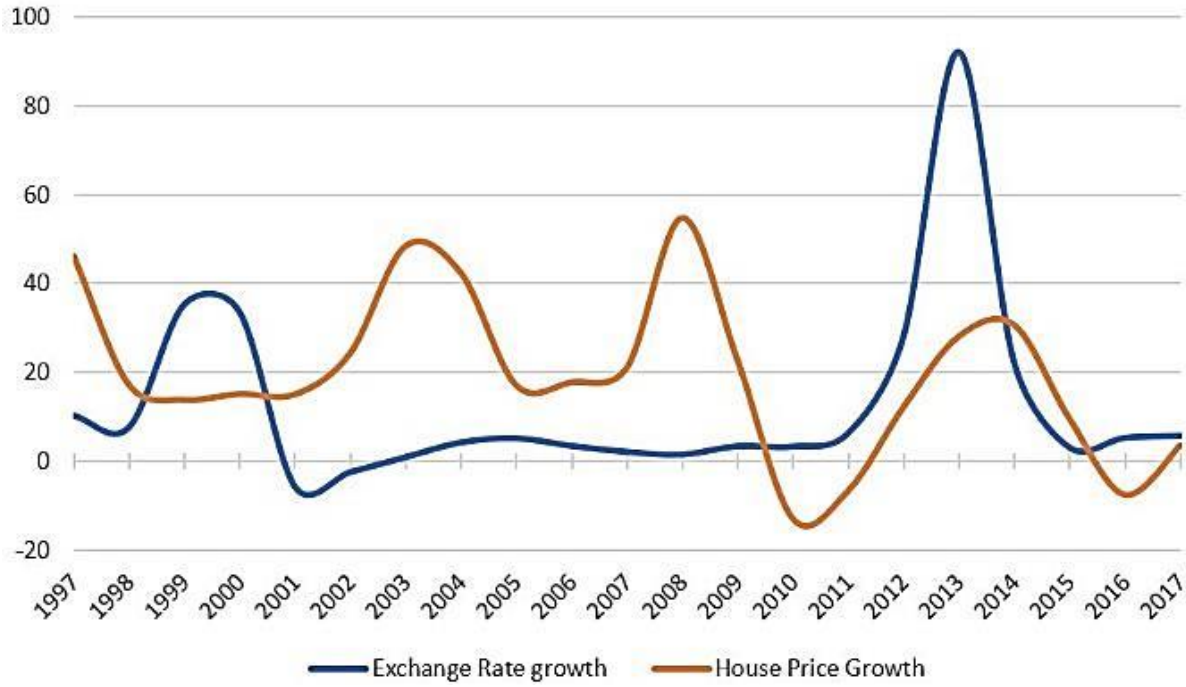

Figure 12. Changes in the Foreign Exchange Rate Impact on the Housing Market
Source: Research finding.

Figure 12 shows that despite the sharp rise in the exchange rate, due to its effect on the increase in the general level of prices and construction costs, the housing sector has faced price growth. According to theoretical literature, due to the normality of housing assets, increasing household per capita income increases the demand for housing, whether consuming demand or capital.

Table 1. Spearman Correlation Coefficient between House Price and Selected Indices

Index Name	rho	p-value
Liquidity Growth	0.4647	0.0405
Real Liquidity Growth	0.5017*	0.0303
Land Price Growth	0.7675	0.0000
New Construction Investment	0.7831	0.0000
Inflation Rate	0.4887	0.0246
Building Material Price Growth	0.4779	0.0299
Construction Service Index	0.4377	0.0485

Source: Research finding (2020).
Note: * A two-year lag has been considered for computing the spearman rank between these two variables.

On the other hand, with the increase in people's income, the average desire for savings increases, and part of this savings can flow to the housing market, which leads to a rise in housing demand and, consequently, its price.

Methodology

In modern pricing models, the price of financial assets is continuous over time, and prices follow the diffusion process. In order to model the price of financial assets in an optimal manner, in a first step, the impact of economic, physical and behavioral factors on financial asset prices should be expressed in the form of diffusion equations, and in the next step, the diffusion equation coefficients should be estimated. By reducing the bias of the coefficients of the diffusion equation, the pricing error will also be reduced. Therefore, the approach used to estimate the diffusion equation coefficients is also essential in reducing the pricing error of financial assets, especially financial derivatives. The geometric Brownian motion model, GBM, and the GBM model with stochastic fluctuations are two common differential equations for modeling the behavior of financial asset prices. There are two basic forms of GBM. First, in this model, the fluctuations of financial asset returns are constant over time, which the empirical results do not confirm the constant volatility. Second, in this model, the returns on assets represent a normal distribution, while empirical results are indicating that the distribution of returns is not normal. In Brownian geometric motion with stochastic fluctuations, although fluctuations are stochastic, the pattern does not usually focus on the break of non-conditional variance. Large shocks always lead to a break in non-conditional variance, so estimating the coefficients of the GARCH model, regardless of the break of the coefficient fluctuation, results in the bias of estimated coefficients and the measurement error of the conditional variance. In stochastic differential equations, two kinds of stochastic nature can be considered; a group of these equations has a differentiable solution, and another group is differentiable.

Each of these two groups provides solutions that are fundamentally different from each other. The first group has more straightforward solutions and includes ordinary differential equations with stochastic coefficients or random initial values or random inputs with regular and specific properties or even a combination of these states. For example, the following linear differential equation:

$$x(t) = \frac{dx(t)}{dt} = a(\omega)x(t) + b(t,\omega) \tag{1}$$

Where, b is a random input and for every ω is continuous with respect to time. This equation with an initial value $x_0 = \omega$ at t = 0 has the following solution:

$$(t,\omega) = e^{a(\omega)t}\left(x_0(\omega) + \int_0^t e^{-a(\omega)s}\, b(t,\omega)ds\right) \tag{2}$$

As can be seen, the solutions obtained from solving this equation are differentiable functions with respect to time t. The second class or group is the equations whose inputs are an irregular stochastic process such as Gaussian white noise. These equations are considered as stochastic differential equations and are expressed in terms of equations with stochastic Ito integrals. Such equations do not have a differentiable solution. The Ito process is an example of this group. Stochastic models have two main parts. The first part represents the mean of variations or drifts at any moment, and the other, models the momentum fluctuations of the process:

$$x(t,\omega) = e^{a(\omega)t}\left(x_0(\omega) + \int_0^t e^{-a(\omega)s}\, b(t,\omega)ds\right) \tag{3}$$

In most models, the drift component is obtained using numerical methods to match the initial value, while there is also an analytical relationship for a small number of models. In modeling using stochastic differential equations, the behavior of the process studied is considered as an Ito process. In this research, the housing price is considered as an Ito process. An Ito process $x = \{x(t) ؛ t \geq o\}$ is of the following general form:

$$x(t) = x(0) + \int_0^t a\big(s, x(s)\big)ds + \int_0^t b(s, x(s)d\omega(s) ؛ t \geq 0 \tag{4}$$

This relation contains the initial value $x(to) = x_0$, which may be stochastic, has a time-variant component with slow changes called drift, and a continuous stochastic component with rapid changes with time called diffusion.

The second integral in the above equation is a stochastic Ito integral with respect to the Weiner process ω $= \{\omega(t) ؛ t \geq 0\}$. The above integral equation is often written in the form of a differential equation as below:

$$x(t) = x(0) + \int_0^t a\big(s, x(s)\big)ds + \int_0^t b(s, x(s)d\omega(s) ؛ t \geq 0 \tag{5}$$

The above equation is called the stochastic differential equation. The solution of the stochastic differential equation is generally complex, and there are only certain cases the solutions of which are known. Despite the general solutions presented by researchers, the distance between the theoretical advances of stochastic differential equations with its practical application is enormous. The Wiener process ω(t) is not differentiable at any point. Therefore, the second term in the last integral equation is not such as a Lebesgue integral or Riemann integral, and the relations of equations relating to its integration are different from the usual integrals so that:

$$\int_0^t W(s, \omega)dW(s, \omega) = \frac{1}{2}W^2(t, \omega) - \frac{1}{2}t \tag{6}$$

The most significant property of a Wiener process is $E[d\omega(t))^2] = dt$ .However, for any $t \geq t_0$, the stochastic process Y(t) can be defined as follows and differentiated:

$$Y(t, \omega) = U(t, x(t, \omega)) \tag{7}$$

Where, $U(t, x)$ is a continuous function with a partial derivative of the second-order, and x (t) is also the Ito integral, in fact $x(t, \omega) = \int_{t_0}^t f(s, \omega)dW(s, \omega)$, or as a differential equation:

$$dx(t) = f(t, \omega)d\omega(t) \tag{8}$$

Using the chain rule, we can differentiate Y (t) as follows:

$$\begin{aligned}
&dY(t) = \tfrac{\partial u}{\partial t}\big(t, x(t)\big)dt + \tfrac{\partial u}{\partial x}(t, x(t)dx(t) \\
&\Delta Y(t) = u\big(t + \Delta t, x(t) + \Delta x(t)\big) - u\big(t, x(t)\big) = \left\{\tfrac{\partial u}{\partial t}\Delta t + \tfrac{\partial u}{\partial t}\Delta x\right\}z + \\
&\tfrac{1}{2}\left\{\tfrac{\partial^2 u}{(\partial t^2)(\Delta t)^2} + 2\tfrac{\partial^2 u}{\partial t \partial x}\Delta t \Delta x + \tfrac{\partial^2 u}{(\partial x^2)(\Delta x)^2}\right\} + O \\
&dY(t) = \tfrac{\partial u}{\partial t}\big(t, x(t)\big)dt + \tfrac{\partial u}{\partial x}(t, x(t)dx(t) + \tfrac{1}{2}\tfrac{\partial^2 u}{\partial x^2}(t, x(t)dx^2(t)
\end{aligned} \tag{9}$$

This relation is the chain rule in stochastic differential equations and is called the Ito formula. According to Equation (9), there is an additional term that is not found in the ordinary chain rule. In this research, based on the above theoretical foundations, three models of geometric Brownian model, Merton model, and Merton model based on nonlinear GARCH model are investigated.

The Method of Research and Estimation of Stochastic Differential Equations for the Housing Price

One of the methods for modeling the housing price is the use of random walk theory. The randomness of the housing price changes does not mean that the price ratio is irrational. If prices are reasonable, only new information will change the price. Therefore, the random walk pattern shows that prices reflect all relevant information. Instead, if the price changes are unpredictable, the market will not be efficient. A random walk is converted to the Brownian process in continuous mode, so a stochastic process can be used to model the housing price. Brownian geometric motion or Black-Scholes model explains the stochastic behavior of the housing price over time. The Black-Scholes model of housing price in the form of a stochastic differential equation is as follows:

$$ds_t = \mu(s.t)\, s_t + \sigma(s.t) s_t dW \tag{10}$$

In Equation (10), the parameter s_t is the housing price at time t, W is the standard Brownian motion, μ and σ are the coefficients of the model. The standard Brownian motion is a diffusion process with a mean of zero and variance σ. Brownian process has a normal distribution while it is also continuous over time, and its changes are independent. As discussed in the theoretical foundations, Ito's relationship is used to solve Equation (10). The solution of the above differential equation is as follows:

$$dlogs_t = \left(\mu - \frac{1}{2}\,\sigma_t^2\right) dt + \sigma(s\,.t)dW \tag{11}$$

According to Eq. (11), the differential of the housing price has a normal distribution with the mean $\mu - \frac{1}{2}\,\sigma_t^2$ and the variance σ_t^2. By taking the integral in the interval from zero to T from the two sides of the above relation and taking antilogarithm of the solution, we get Equation (12):

$$S(T) = S(0)^{\left(\mu - \frac{1}{2}\sigma_t^2\right)T + \sigma W(T)} \tag{12}$$

Table 2. GMB of Housing Price

$ds_t^h = \mu\, s_t^h\, dt + \sigma\, s_t^h\, dW$				
Parameter	**Coefficient**	**S.D**	**Z**	$\boldsymbol{P > \lvert Z \rvert}$
M	0.021	0.004	4.26	0.000
σ	0.002	0.0003	7.31	0.000
Log-likelihood=165.86				

Source: Research finding (2020).

According to this relationship, the housing price has a normal distribution. Given the definite distribution of the housing price, a maximum likelihood approach is applicable for estimating the coefficients of the differential equation. One of the features of the standard Brownian relationship is to follow from the Markov process. Thus, we can write the likelihood function of the standard Brownian relationship as the product of the likelihood density. In **Table 2**, the results of estimating the coefficients of geometric Brownian motion patterns are presented in the form of a stochastic differential equation for the housing price. According to the results shown in **Table 2**, the value of the drift parameter (μ) for the housing price index is

0.021, and the amount of diffusion parameter (σ) is 0.002. In the pattern of Brownian geometric motion with jump (Merton pattern), in addition to the two parameters of drift (μ) and diffusion (σ), the number of jumps (*lambda*), mean, and variance of the jump, *delta* and *Muj* are added, respectively.

According to the results of Table 3, the probability of jump in the price ratio in the housing price in the market is 1.73. Also, the average jump size of the housing price is 0.101, and the jump variance of housing prices is 0.0049. It suggests that there is no market-efficient hypothesis in Iran's housing market because the presence of sudden jump under systematic risks is indicative of an increase in inefficiencies in the housing market.

One of the disadvantages of the geometric Brownian motion pattern is to consider fluctuation to be fixed, and another problem with this pattern is the lack of attention to the role of information in the markets. However, the results of empirical studies indicate the randomness of fluctuations in returns. In other words, the changes in returns follow the GARCH process. Also, information available in the market is crucial in explaining the behavior of the housing price. Using the GARCH models and modeling the behavior of fluctuations, the above problem can be solved. In order to model the behavior of the housing price, considering the asymmetric effects of good and bad news, a nonlinear GARCH model was used to explain the behavior of price fluctuations over time.

In the output presented in Table 3 in addition to the mean values of the Geometric Brownian Motion(GBM) pattern, the new information on the geometric Brownian motion pattern along with the nonlinear GARCH, namely, the coefficient of the squared error (beta), the coefficient of conditional variance (alpha) and parameter Gamma that shows the impact of negative and positive shocks are presented. Additionally, omega is the constant value of the nonlinear GARCH model.

Table 3. Merton GMB with Involving the Jump in Housing Price

$ds_t^{TSE} = \mu s_t^{TSE} d_t + \sigma s_t^{TSE} dW + dq_t$

Parameter	Coefficient	S.D	Z	$P > \|Z\|$
μ	0.061	0.061	1.1	0.27
σ	0.012	0.003	4.6	0.000
Lambda	1.730	0.650	2.62	0.009
Delta	0.0049	0.0006	-0.78	0.43
Muj	0.101	0.010	8.94	0.000
Log-likelihood=173.76				

Source: Research finding (2020).

The results of estimating the following model are shown in
Table **4**. According to the results of the non-linear GARCH (NGARCH) model, the estimated coefficient γ in the market is positive. It means that the housing price is more affected by bad news and adverse shocks. The numerical value of the coefficient γ in the housing market is 3.15, indicating that it is least affected by positive news. In total, according to the estimated equations in Iran's housing market and the maximum likelihood function, the geometric Brownian model with stochastic NGARCH-based fluctuations has more explanatory power than the Merton model and the geometric Brownian model with constant volatility.

Table 4. Merton GMB with Involving the Jump in Housing Price

$s(t_i) = \mu(t_i)sdt_i + \sigma(t_i)sdW(t_i)$

$\sigma(t_i)^2 = \omega + \alpha\sigma(t_{i-1})^2 + \beta(\varepsilon(t_{i-1}) - \gamma\sigma(t_{i-1}))^2$

$\varepsilon(t_{i-1})^2 = \sigma(t_i)dW(t_i)$

$ds_t^h = \mu s_t^h dt + \sigma s_t^h dW + dq_t$

Parameter	Coefficient	S.D	z	$P > \|Z\|$

μ	0.013	0.005	2.46	0.014
ω	0.001	0.0004	2.43	0.015
α	1.02	0.99	1.03	0.3
β	-0.04	0.09	-0.48	0.63
Γ	3.15	7.11	0.44	0.65
Log-likelihood=179.70				

Source: Research finding (2020).

Conclusion and Policy Recommendations

The main objective of this study was to examine the behavior of the housing price in Iran based on stochastic differential equations.

Accordingly, the Black-Scholes model, the Merton model, and the non-linear GARCH (NGARCH) model based on the Merton model have been used. The study of drift coefficients and diffusion coefficients in the estimated equations implies that the housing price in Iran has a stochastic nature. The fluctuations of this market and the estimated jump factors in this study indicate that the efficient-market hypothesis does not hold in the Iran housing market. This suggests that housing policy-making in Iran has not been well suited to determine an appropriate model for the rules of the game in this market, and the coefficient of failure in policy-making in achieving an efficient market is evident on the basis of estimated differential equations. According to the non-linear GARCH model based on the Merton model, evidence implies the asymmetric behavior of the players in this market. So that the response of the actors to the negative and positive shocks was very different. According to the findings and the inefficiency of the Merton model in the Iranian housing market, it is undoubtedly necessary to adjust and correct the housing price as a central and influential variable on the economy of the country. However, this adjustment and correction should be made within the framework of a corrective system and a specific modification table. If the changes in the housing price are made separately and regardless of this key point and other realities ruling the economy of the country, it may result in undesirable outcomes, and may even lead to violations of goals and revers objectives. If it is possible to regulate and direct the mechanisms of the country's economy in such a way that lead to the creation of a common housing system, the adjustment of the housing price would automatically be carried out, and the complementary and necessary modifications in other sectors would be realized endogenously.

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