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RESEARCH PAPER

The Extension of the Spokes Model to the Streets with Unequal Length Model (Explosion Model)

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Abstract
Locating is one of the key elements in the success and survival of industrial centers and greatly affects cost reduction of the establishment and launching of various economic activities. This study has used the streets with unequal-length model through the classic extension of the Spoke model (Lijesen and Reggiani, 2013), but with an unlimited number of streets with uneven lengths. The results show that the spoke model is a special case of the streets with unequal-length model. According to the results of this study, if the strategy of enterprises and firms is to select both price and location, there would be no balance in the game. Furthermore, the increased length of streets leads to an increased profit of enterprises, while the enterprises choose locations far from the center (the maximum differentiation) with the increased number of streets, consequently decreasing the enterprises' output. Moreover, the enterprise production rate will incline toward zero when the number of streets goes to infinity, and a complete competition outcome will be achieved.
Keywords: Locating, Nash Equilibrium, Streets with Equal-Length Model, Streets with Unequal-Length Model.
JEL Classification: C62, D43, L11.

1. Introduction
The spatial dimension is one of the main issues in individuals and enterprises' life and decision-making in the economy. Meanwhile, policy-making on the establishment of industries, without any knowledge, leads to the elimination of or decline in the efficiency of the economic systems, highlighting the significance of investigating the locating issue (Vinay and Chakra, 2005). With the expansion of the capitalist economic approach and consideration of location as an economic commodity, competition for establishment in the most appropriate location has intensified, and the selection of the best location has been considered the main objective of economic activities. Banks are not the only entities concerned with the

issue of location, as consumers are also seeking to decrease their costs by reducing transportation costs. Previous studies have found that location parameters vary with distance from the city center and between affluent and disadvantaged neighborhoods (Gelormino et al., 2015; Zhao et al., 2018). Also, the growing concern about the significant increase in transportation costs has led theorists to endorse the idea that both housing and transportation affordability are critical rather than housing affordability alone (Jetzkowitz et al., 2007; Deka, 2015, Haas et al., 2013).

Locating studies are one of the key actions in the construction process of industrial or service units, contributing significantly to the success of the centers. The importance of such studies is to the extent that they have been recently reconsidered for active centers, sometimes leading to changes in the location of the industrial units. The economic success of an industrial enterprise depends not only on its technical and economic efficiency but also on the location of the intended activity; thus, the theories of locating have been proposed, along with the development of industries and their social and economic consequences, to ensure higher productivity of industrial activities and reduce negative effects and economic losses. On the other hand, consumers' poor access in fringe areas often leads to a reliance on private vehicles for daily commuting, adding a significant capital expenditure to the household budget and increasing daily transportation costs (Currie and Senbergs, 2007). This not only raises the daily transportation cost but also increases the household's total expenditure-to-income ratio (Viggers and Howden-Chapman, 2011). Also, with urbanization on the rise, housing and transportation policies must strive to strike a balance between accessibility and affordability (Marval and Silva, 2023).

The location could be interpreted in two ways, referring to the physical location of a certain consumer or representing the distance between the desired characteristics of the trademark with the characteristics of practically purchased one (Shy, 2014). Melaniphy (1999) showed that 50% of quick-return occupations would be bankrupted in the first year and about 30% after two years, turning into another occupation. Although all aspects of service provision are considered at the start-up of these businesses, ignorance of the important issue of location would prevent the enterprises from achieving the intended profit. The locating models are divided into competitive and non-competitive categories. Competitive locating is the main aspect of locating issues used in the game theory where the competing enterprises seek to maximize their market share. The primary issue in these situations is the optimum locating of one or several new facilities in the market where other competitors have been previously present. By modeling the problem in form of a game, it is possible to obtain optimum location with Nash equilibrium while considering the location of the existing facilities and decision-making about the enterprises that will likely enter the market in the future. In this case, the rival enterprises try to attract customers with the strategic decisions that they select, necessitating familiarity with different types of customers' behavior.

The oldest location model dates back to Saffle in 1878. Thinkers such as Lanhart (1882) and Alfred Veber (1909) completed and proposed new location models by World War II. Competitive positioning models with a game theory approach were first introduced by Hotelling in 1929. However, further research was needed to improve one or more of the assumptions of the Hotelling model and to make it more general (such as Salop, 1979, Chen and Riordan, 2007, etc.). The most significant point in these generalizations is that competitive location models are inherently unstable. In other words, completely different results would be obtained with the slightest change in an assumption or a parameter. It should be noted that few theoretical studies have examined modeling of location models of firms; therefore, some studies focusing on the locating of sale agents will be reviewed in the following:

Anderson and Neven (1986), Vandenbousch and Weinberg (1995), Yang and Lee (1997), Michael et al. (1998), Burdurla and Ejder (2003), Choo and Mazzrol (2003), Rhim et al. (2003), Bautista and Pereira (2007), Redondo et al. (2008), Reggiani (2009), Liu et al. (2010), Choo et al. (2010), Wongsak (2011), Lu et al. (2012), Miquer et al. (2014), Tartavulea (2015), Hu and Wang (2017), Dewita et al. (2020), and Harrison and Campbell (2021) conducted experimental research on the location of firms.

Chen and Riordan (2007) considered a market consisting of N streets of equal length with the same center and the length of each street model as 1/2. They assumed that the consumers on each street were equal and uniformly distributed on streets such that there were $\frac{2}{N}$ consumers on each street. They showed that the entry of a new enterprise would change the consumer surplus and social welfare through price, market development, and their comparative impacts. With free entrance of enterprises, the total production of market might be less or more compared to social optimum, and the equilibrium price will be more than the final cost when the number of enterprises is high.

Lijesen and Reggiani (2013; 2016) considered the spoke model of Chen and Riordan, indicating that the optimum location of enterprises depended on whether the consumers were confused in equal streets or not. If they were involved and confused, the market power effect was the determining factor, and enterprises increased their distinctions while also increasing the communication between the market divisions. In this case of price competition, the effect of market share is determining, and enterprises are inclined to become close to each other.

In a research titled expansion of location theories of firms and products' consistency using triangular distribution approach, Shahbazi and Salimian (2017) addressed the issue of location of companies using triangular distribution density function. They stated that location models usually use a uniform distribution of consumers, while this is not the case in reality, and consumers are more concentrated in city centers than in suburban areas. They solved the spoke model of Lijesen and Reggiani (2013) by changing the distribution of consumers from uniform to triangular in a two-stage game. Their results showed that the increased number of streets and transportation costs would lead to price increase, indicating that as the firms get farther away from each other, their competition in the market would decrease, and the price would increase. If both firms are located in the same distance from the city center, they would gain the same market share and more inclined to be closer to or have the minimum distance from the city center.

Baraklianos et al. (2020) sought to find out whether accessibility criteria would affect the results of residential location choice modeling. They tested a residential location selection model for the urban area of Lyon in France with different accessibility indicators and showed that accessibility was an essential variable. Ahrens and Lyons (2021) presented a gravity model of commuting flows in Ireland to address the question of whether an increase in rent would lead to longer commuting. They stated that some commuters might be forced to take on longer commutes due to rising rents in central locations. Thus, they considered a gravity model of commuting flows for Ireland over 2011-2016, indicating that a 10% rise in rents in employment centers was associated with an up to 0.6-minute rise in one-way daily average commuting times nationally (about 2.2% of the average commute duration).

Marval and Silva (2023) investigated the residential location in a research titled city affordability and residential location choice: A demonstration using an agent-based model. They stated that with urbanization on the rise, housing and transportation policies must strive to strike a balance between accessibility and affordability. Their study built an economic rational agent-based model for a hypothetical monocentric city to simulate the urban pattern that emerged from households' residential location choice, as they aimed to minimize their expenditure on rent and commute under different scenarios. The model highlighted the significance of housing and transportation costs as a spatial policy tool in shaping urban growth. It was also shown that private transportation users tended to reside in the city's inner areas, while public transportation users opted for outer areas. However, when public transportation was heavily subsidized, this pattern was reversed.

Salimian et al. (2023) investigated the optimal location of sales agents and their optimal number in a research titled locating the sales agents in the spoke model through uniform distribution of consumers. The main objective of their research was the theoretical modeling of sales agents and expansion of location models through a method, in which the assumptions were closer to reality and could provide the required conditions for the selection of the optimal location and optimal number of sales agents. The results highlighted the conditions under which the city center or margin was the optimal location of sales agents, indicating that the cost of launching sales agents was the main factor in making such decisions. Moreover, the results showed that the optimal number of sales agents was a function of the number of streets, the customers' valuation of each unit of product, the price of sale agents, the number of consumers on each street, the earned profit by the sales agents, and the cost of launching sales agents.

The locating problem has been considered since the 1960s and divided into competitive and non-competitive categories in a classification such that the share of non-competitive models was much higher than competitive ones. The competitive locating model with game theory approach was introduced first by Hotelling concerning the competition between two ice cream vendors in 1929. The next studies focused on improving one or several assumptions of Hotelling model hypotheses at a more general level. The main point in these generalizations was the inherent instability of the competitive location models. In other words, with the lowest changes in an assumption or a parameter, totally different results would be obtained. After Hotelling (1979), Salop developed the circular city model, and in the following years, the Spock model was presented by Chen and Riordan in 2007, and further studied by Lijesen and Reggiani in 2013 and 2016. However, these studies examined only one or more of Hotelling assumptions to modify and improve the model. Therefore, over the years, locating models have reached the spoke model (streets of equal length), the main assumption of which (using unequal-length streets) is investigated and modified in this research. Also, the results have been compared with the previous models to verify the validity of the assumptions and the model presented in this research. It should be noted that this model was presented and named by the authors of this research for the first time. Also, the assumptions of this model cover all the previous models. For example, the spoke model is obtained if the length of the streets is equal, the Hotelling model is achieved if one street is considered, etc.

On the other hand, the main problem of these models has been in simplifying the assumptions. One of these models, presented by Chen and Riordan (2007) and extended by Lijesen and Reggiani (2013; 2016), was the spoke model, which was the generalized form of the circular city model, considering N streets of equal lengths of $\frac{1}{2}$. All these streets were connected to the city center, and in case that the enterprise was not located in the street of equal length as of the consumer or the consumer was on an empty street, the consumer had to pass through the city center to purchase from that enterprise.

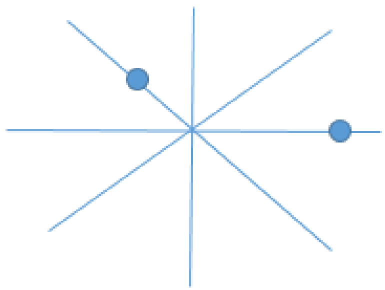

Figure 1. The Spokes Model, 2 Firms, N = 8 Spokes each Length of $\frac{1}{2}$

Source: Research finding.

These assumptions are just for the simplification of the model and its results. Here, all assumptions except for equality of the length of streets are considered fixed, which is one of the main and even the most critical simplification assumptions; however, the distance of all streets up to the city center is not equal in real world. This model is called the streets with unequal-length model (Explosion Model)[1].

Since the real world consists of cities with unequal streets lengths, this section seeks to tackle the weaknesses of Lijesen and Reggiani work and propose a more appropriate solution in enterprises locating. Concerning the above-mentioned points, this study primarily aims to extend and generalize the competition in the spoke model to streets with the unequal-length model through the following research questions: How will the results of the spoke model change for the streets passing the

[1]. The reason for selecting this name for the considered model is that all particles will not necessarily pass the same distance from the explosion center in an explosion. In other words, every particle will be at different distance from the explosion center such as big bang theory.

center are of different lengths? Are the results of the spoke model true for the streets with unequal length model? Do enterprises' location and profit change with the change in the length of streets? How does the optimal location of enterprises change in such conditions? Do the enterprises select the farther location to the center as the optimal location or are they inclined to the center? How does the increase in the number of streets affect the location and profit of enterprises?

This paper is structured in 5 sections. The second section has presented the model. The third section focuses on location in the streets with unequal-length model in five subsections. The fourth section has presented selection of price and location, and finally, recommendations and conclusion are presented in the fifth section.

2. Model

This paper has used the spoke model (competitive location choice) proposed by Chen and Riordan (2007) and studied by Lijesen and Reggiani (2013; 2016), with the difference that the length of streets is not equal, or more specifically, not equal to $\frac{1}{2}$ here. The market is consisted of N streets of unequal lengths, and any street has a length of l_i (i=1, …., N). The consumers are uniformly distributed as $\frac{2}{N}$ on every street. Following Chen and Riordan (2007), it is assumed that the evaluation of any consumer from desirability obtained from products' purchase is of rate *v*. It is assumed that the products are offered by two enterprises potentially located on the same street that the consumer is located or on any other of N streets. The street on which the consumer is located is introduced by s and shown by l_s. In addition, the enterprises cannot trespass on other streets. In the end, the strategy of enterprises is as y_i ε $(-\infty, l_i)$, i= 1, 2.

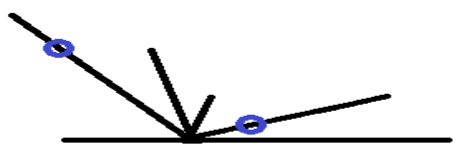

Figure 2. Streets with Unequal-Length Model with six Streets of Unequal Lengths and Two Enterprises Shown on them
Source: Research finding.

Figure 2 shows the streets with unequal-length model with the highlighted points showing the situation of enterprises. The customer's situation on street s is shown by x_s and defined as $x_s \varepsilon [0, 1]$. The distance between the consumer located on streets s (x_s) and the enterprise located on the street i (y_i) is shown by $d\ (y_i, x_s)$, defined based on whether the consumer and enterprise are located on the same or different streets. If they are both located on street i, then:

$$d\ (y_i, x_s) = |\ y_i - x_s\ | \qquad s = i \tag{1}$$

On the other hand, if the enterprises are located on different streets, we will have:

$$d\ (y_i, x_s) = (l_i - y_i) + (l_s - x_s) = l_i + l_s - y_i - x_s \quad \forall\ s \neq i \tag{2}$$

In addition, it is assumed that in case the consumer purchases from other enterprises located in other streets, they are obliged to pass through the city center. Following Lijesen and Reggiani (2013; 2016), it is assumed that transportation costs are a ratio of square distance between the consumer and enterprise. In other words, transportation costs are defined as:

$$T_{is}\ (y_i, x_s) = \begin{cases} td^2(y_i, x_s) = t\ (y_i - x_s)^2 & s = i \\ td^2(y_i, x_s) = t\ (l_i + l_s - y_i - x_s)^2 & \forall\ s \neq i \end{cases} \tag{3}$$

Without losing the whole issue and for simplification, it is assumed that t=1, and the final production cost in all analysis procedures is zero (c=0). The enterprises play a three-step game with incomplete information:

1. The nature allocates a street s=i to each enterprise i= 1, 2.
2. Enterprises simultaneously determine their location y_i on the street they are located.
3. Enterprises simultaneously demand a uniform price p_i (Lijesen and Reggiani, 2013).

In this section, the game will be solved through the backward induction method.

3. Location in the Streets with Unequal-Length Model

We consider a two-round game where the enterprises decide to determine their location in the first round and the price in the second round. To solve this game, we begin from the pricing stage and define our profit function. The first step to determine the profit function is to determine the indifferent consumers. Following Chen and Riordan (2007) and Lijesen and Reggiani (2013; 2016), it is assumed that each consumer has preferences for just two signs, and there will be no problem with whether there is an enterprise in the market or not (Lijesen and Reggiani, 2013, 2016). This assumption indicates that the consumer located in each street prefers the products produced by the enterprise located on his street and as the second desired sign, he might prefer each sign of N-1 other streets (the producing sign of enterprise

2, or N-2 other enterprises that don't supply) (Lijesen and Reggiani, 2013 and 2016). Thus, there are two types of consumers, including those who prefer both existing signs and those just preferring one type of the presented signs. Moreover, some consumers prefer two signs that are not offered by any enterprise, indicating that the whole market is not completely covered. Thus, the enterprises' competition is on type 1 consumers, who are indifferent to purchase from each *i* and *j* enterprise whenever:

$$x_{ij} \ \ s.t. \ \ T_{is}(y_i, x_{ij}) + p_i = T_{js}(y_j, x_{ij}) + p_j \tag{4}$$

Thus:

$$x_{ij} = \frac{1}{2}(l_i + l_s) - \frac{1}{2}(y_i - y_j) - \frac{p_j - p_i}{2(l_i + l_s - y_i - y_j)} \qquad x_{ij} \in j \tag{5}$$

$$x_{ij} = \frac{1}{2}(l_i + l_s) + \frac{1}{2}(y_i - y_j) - \frac{p_j - p_i}{2(l_i + l_s - y_i - y_j)} \qquad x_{ij} \in i \tag{6}$$

Following Lijesen and Reggiani (2013; 2016), the first area of Chen and Riordan (2007) is focused, where v is sufficiently high so that all consumers could purchase on the streets and $x_{ij} = 1$, i = 1,2, j = 3…N. Consequently, other enterprises will achieve equilibrium price by having a location.

3.1 Price Selection

Assuming that the location of enterprise *i* is in y_i, we will begin from cost below the game cost. The profit function of enterprise 1 will be as follows with respect to all consumers who could purchase from there:

$$\pi_1 = \frac{2}{N}\left(\frac{1}{N-1}p_1(1 - x_{12}) + \frac{1}{N-1}\sum_{j=3}^{N} p_1 x_{1j}\right) \tag{7}$$

In the above equation, $\frac{2}{N}$ is the distribution of consumers on each street and $\frac{1}{N-1}$ is the ratio of consumers who have the second desired sign and located on each of *S* streets. X_{12} shows the indifferent consumer in purchasing from each enterprise 1 and 2, and X_{1j} is the indifferent consumer in purchasing from enterprises 1 and j, where the number of enterprises j varies from 3 to N (all enterprises except 1 and 2). After simplification and assuming that the indifferent consumer in purchasing from enterprises 1 and 2 (X_{12}) is located on street 2, the profit function of enterprise 1 will be achieved as follows:

$$\pi_1 = \frac{2}{N}\left(\frac{1}{N-1}p_1\left(1 - \left(\frac{1}{2}(l_1 + l_2) - \frac{1}{2}(y_1 - y_2) - \frac{p_2 - p_1}{2(l_1 + l_2 - y_1 - y_2)}\right)\right) + \frac{N-2}{N-1}p_1\right) \tag{8}$$

The following results could be obtained from the profit functions:

Theorem 1: *There is a unit Nash equilibrium in prices, and this equilibrium price is increasing in N.*

Proof: To prove the theorem, we need to derive the first and second derivations from the profit function relative to prices:

$$\frac{-2l_1-2l_2+2y_1+2y_2-l_1^2-2l_1l_2+2l_1y_1-l_2^2+2l_2y_1-y_1^2+y_2^2+p_2-2p_1+2N(l_1+l_2-y_1-y_2)}{N(N-1)(y_1+y_2-l_1-l_2)}=0 \quad \frac{\partial \pi_1}{\partial p_1}=-$$

$$\frac{\partial^2 \pi_1}{\partial p_1^2}=\frac{2}{N(N-1)(y_1+y_2-l_1-l_2)}<0 \quad if \quad y_1+y_2<l_1+l_2$$

Since $y_1+y_2<l_1+l_2$ is always established (except in certain conditions when both enterprises are located at the city center), the enterprises are never established in the city center. Considering symmetry, it is possible to easily calculate P_1 and P_2 values.

$$+\frac{1}{2}y_2^2-\Big(l_1+l_2+p_1=\frac{1}{2}p_2+y_1(1+l_1+l_2)+y_2+N(l_1+l_2-y_1-y_2)-\frac{1}{2}y_1^2$$
$$\frac{1}{2}l_1^2+\frac{1}{2}l_2^2+l_1l_2\Big)$$

$$-\frac{1}{2}y_2^2-\Big(l_1+l_2+p_2=\frac{1}{2}p_1+y_1+y_2(1+l_1+l_2)+N(l_1+l_2-y_1-y_2)+\frac{1}{2}y_1^2$$
$$\frac{1}{2}l_1^2+\frac{1}{2}l_2^2+l_1l_2\Big)$$

Solving the above equation, P_1 and P_2 Nash equilibrium values will be achieved as follows:

$$p_1=\frac{4}{3}\left(\frac{3}{2}+l_1+l_2\right)y_1+\frac{4}{3}\left(\frac{3+l_1+l_2}{2}\right)y_2+2N(l_1+l_2-y_1-y_2)-\frac{y_1^2}{3}+\frac{y_2^2}{3}-2\left(l_1+l_2+\frac{1}{2}l_1^2+\frac{1}{2}l_2^2+l_1l_2\right) \tag{9}$$

$$p_2=\frac{4}{3}\left(\frac{3+l_1+l_2}{2}\right)y_1+\frac{4}{3}\left(\frac{3}{2}+l_1+l_2\right)y_2+2N(l_1+l_2-y_1-y_2)+\frac{y_1^2}{3}-\frac{y_2^2}{3}-2\left(l_1+l_2+\frac{1}{2}l_1^2+\frac{1}{2}l_2^2+l_1l_2\right) \tag{10}$$

From Equations 9 and 10, we have:

$$\frac{\partial p_1}{\partial N}=2(l_1+l_2-y_1-y_2)=\frac{\partial p_2}{\partial N}$$
$$\Rightarrow l_1+l_2-y_1-y_2>0 \;\; if \;\; l_1+l_2>y_1+y_2 \;\; or \;\; y_1<(l_1+l_2)-y_2 \tag{11}$$

Since:

$$\frac{\partial p_1}{\partial N}=\frac{\partial p_2}{\partial N}\geq 0$$

Notably, $l_1+l_2>y_1+y_2$ is always established, as enterprises 1 and 2 with locations y_1 and y_2 are on the same streets with unequal lengths of l_1 and l_2, established at the ending point of these streets that is city center (where the equality will be established).

The results obtained by replacing $l_1 = l_2 = \frac{1}{2}$ values are similar to the spoke model of Lijesen and Reggiani (2013; 2016).

Theorem 2: *If the following conditions are established:*

$y_1 < -3N + 2l_1 + 2l_2 + 3$ ، $y_1 > 3N - l_1 - l_2 - 3$

$y_2 < -3N + 2l_1 + 2l_2 + 3$ ، $y_2 > 3N - l_1 - l_2 - 3$

An increase in y1 and y2 leads to the increased price of enterprises 1 and 2 and vice versa.

Proof: By taking derivations of equations 9 and 10, we have:

$$\frac{\partial p_1}{\partial y_1} = \frac{4}{3}\left(\frac{3}{2} + l_1 + l_2\right) - 2N - \frac{2}{3}y_1 > 0 \Leftrightarrow y_1 < -3N + 2l_1 + 2l_2 + 3 \quad (12)$$

$$\frac{\partial p_1}{\partial y_2} = \frac{4}{3}\left(\frac{(3 + l_1 + l_2)}{2}\right) - 2N + \frac{2}{3}y_2 > 0 \Leftrightarrow y_2 > 3N - l_1 - l_2 - 3 \quad (13)$$

$$\frac{\partial p_2}{\partial y_1} = \frac{4}{3}\left(\frac{(3 + l_1 + l_2)}{2}\right) - 2N + \frac{2}{3}y_1 > 0 \Leftrightarrow y_1 > 3N - l_1 - l_2 - 3 \quad (14)$$

$$\frac{\partial p_2}{\partial y_2} = \frac{4}{3}\left(\frac{3}{2} + l_1 + l_2\right) - 2N - \frac{2}{3}y_2 > 0 \Leftrightarrow y_2 < -3N + 2l_1 + 2l_2 + 3 \quad (15)$$

These results show that under the above conditions, the farther enterprise 1 becomes from the center, it could increase its price. The price of enterprise 1 also increases when enterprise 2 becomes closer to the center and vice versa. The same condition is true for enterprise 2. In the backward induction method, the second step is extracting equilibrium location (by replacing the obtained equilibrium prices from the previous stage), which will be addressed in the next part. These results are in line with the results obtained by Shahbazi and Salimian (2017). They showed the price of firm1 had a reverse relation with its location, while this relation was direct in Hotelling uniform distribution method, in which the more y_1 increased, the closer this firm became to the middle points, leading to higher prices. On the other hand, in a Bertrand game with differentiated products, the profit of firms increases with differentiation of products. It means that the differentiation of the products increases the monopoly power of firms producing trademarks with decreasing the price competition among the firms producing trademarks (Shy, 1996).

3.2 Location Choice

Concerning the obtained results from price and placing in the profit function of the enterprise (Equation 8), it is possible to determine the situation of enterprises:

$$\pi_1 = \frac{2}{N}\left(\frac{1}{N-1}p_1(y_1, y_2)\left(1 - \left(\frac{1}{2}(l_1 + l_2) - \frac{1}{2}(y_1 - y_2) - \frac{p_2(y_1, y_2) - p_1(y_1, y_2)}{2(l_1 + l_2 - y_1 - y_2)}\right)\right) + \frac{N-2}{N-1}p_1(y_1, y_2)\right) \quad (16)$$

The following results can be extracted from the profit function:

Theorem 3: *There is a complete sub-game of Nash equilibrium in place. This symmetric combination and optimum location is decreasing with respect to N.*

Proof: From the obtained and simplified profit function (Equation 16), after replacing equilibrium prices and the first and second rank conditions, we have:

$$\frac{\partial \pi_1(y_1, y_2)}{\partial y_1} = \frac{1}{9N(1-N)}\Big(3y_1^2 - 2y_1(y_2 - 12N + 7l_1 + 7l_2 + 12) - y_2^2 + 2y_2(l_1 + l_2) + 3(2N - l_1 - l_2 - 2)(6N - 5l_1 - 5l_2 - 6)\Big) = 0 \quad (17)$$

$$\frac{\partial^2 \pi_1(y_1, y_2)}{\partial y_1^2} = \frac{2(3y_1 - y_2 + 12N - 7l_1 - 7l_2 - 12)}{9N(1-N)} < 0 \Leftrightarrow \frac{1}{N(N-1)} > 0 \Rightarrow y_1 > \frac{y_2 - 12N + 7l_1 + 7l_2 + 12}{3} \quad (18)$$

The following results will be achieved for the optimal location of enterprises by solving the above equations and the profit function of enterprise 2 and differentiating and solving it, along with these equations:

1) $y_1 = \frac{l_1 + l_2}{2}$, $y_2 = \frac{-12N + 7l_1 + 7l_2 + 12}{2}$

2) $y_1 = \frac{-12N + 7l_1 + 7l_2 + 12}{2}$, $y_2 = \frac{l_1 + l_2}{2}$ (19)

3) $y_1 = \frac{-6N + 5l_1 + 5l_2 + 6}{4}$, $y_2 = \frac{-6N + 5l_1 + 5l_2 + 6}{4}$

Considering the second rank condition, it is clear that states 1 and 2 show the situations where enterprises 1 and 2 will unilaterally change their locations, respectively. Thus, neither of the first states is Nash equilibrium for enterprises. State 3 shows Nash equilibrium for the location. It is obvious that: $\partial Y_1 / \partial N = \partial Y_2 / \partial N = -6/4$. Thus, the results of the above theorem will be proved. Considering $l_1 = l_2 = 1/2$, these results are consistent with the results obtained by Lijesen and Reggiani (2013). Moreover, considering the special status of N=2, the results of Hotelling standard model will be achieved, in which the enterprises are $[(5l_1 + 5l_2 - 6)/4, (5l_1 + 5l_2 - 6)/4]$, and the results will be as $[-1/4, -1/4]$ in special status of $l_2 = l_1 = 1/2$. Here, the results of Lijesen and Reggiani are again confirmed. In this state, where all streets are covered by enterprises, the optimal location of enterprises is out of the market area (where the consumers are distributed) (Lijesen and Reggiani, 2016).The enterprises have maximum distinction in so far as the equilibrium location is decreasing with respect to N (Lijesen and Reggiani, 2016). Since equilibrium location is decreasing with respect to N, it is concluded that the enterprises have maximum distinction and differentiation. Lijesen and Reggiani (2016) showed that if the consumers were concentrated, the enterprises would be inclined toward concentration. The consumers were supposed to be located on the empty streets at the center; i.e., there were consumers of the same number of empty streets at the center.

3.3 Complete Competition and Location

In this section, all consumers are considered on all streets. Similar to other location models, it is assumed that the consumer is indifferent to the supplier enterprises and purchases from the enterprise with the minimum price and squared transportation costs. The optimization problem for enterprises 1 and 2 will be solved when the number of streets (N) is ≥2. The certain status of N=2 corresponds to the Hotelling standard status. Demand for the streets where the enterprise is located (with $\frac{2}{N}$ consumers on each street) will be defined as follows:

$$x_{12} = \frac{1}{2}(l_1 + l_2) - \frac{1}{2}(y_1 - y_2) - \frac{p_2 - p_1}{2(l_1 + l_2 - y_1 - y_2)} \quad (20)$$

Demand for consumers located on empty streets depends on the spatial combination of enterprises and their symmetric or asymmetric situation. First, the status will be considered where the enterprises are symmetric with respect to the center.

Demand if the Enterprises are Asymmetric: In this state, the only decision-making variable for consumers on empty streets is the commodity price, and the enterprise with the least price will attract all consumers on the empty streets. For the same prices, it is assumed that the consumers on the empty streets are equally distributed between enterprises; thus, we have:

$$q_1 = \frac{2}{N}\left(1 - \frac{1}{2}(l_1 + l_2) + \frac{1}{2}(y_1 - y_2) + \frac{p_2 - p_1}{2(l_1 + l_2 - y_1 - y_2)}\right) + \frac{N-2}{N} \quad if\ p_1 < p_2 \quad (21)$$

$$q_2 = \frac{2}{N}\left(1 - \frac{1}{2}(l_1 + l_2) + \frac{1}{2}(y_2 - y_1) + \frac{p_1 - p_2}{2(l_1 + l_2 - y_1 - y_2)}\right) \quad if\ p_1 < p_2 \quad (22)$$

$$q_1 = q_2 = \frac{2}{N}\left(1 - \frac{1}{2}(l_1 + l_2) + \frac{1}{2}(y_1 - y_2) + \frac{p_2 - p_1}{2(l_1 + l_2 - y_1 - y_2)}\right) + \frac{N-2}{2N}x_{1j} \quad if\ p_1 = p_2 \quad (23)$$

In the full symmetry condition ($p_1 = p_2$, $l_1 - y_1 = l_2 - y_2$), we have:

$$= \frac{1}{Nt\,(y_1 + y_2 - l_1 - l_2)} - \frac{N-2}{2N} = \frac{\partial q_2 \partial q_1}{\partial p_2 \partial p_1}$$

Thus:

Theorem 4: *Assuming we have the number of streets and enterprise locations, an enterprise could increase its production by an increase in the price. Moreover, assuming we have the number of streets, it is possible to determine the location of enterprises, and assuming we have the special location, it is possible to decide on the production increase or decrease having the number of streets.*

Proof: In so far as:

$$\frac{\partial q_1}{\partial p_1} = \frac{1}{Nt\,(y_1 + y_2 - l_1 - l_2)} - \frac{N-2}{2N} = \frac{\partial q_2}{\partial p_2}$$

Solving with respect to y_1:

$$t = 1 \Rightarrow \frac{1}{N\,(y_1 + y_2 - l_1 - l_2)} - \frac{N-2}{2N} > 0 \tag{24}$$

$$if \quad y_1 < (l_1 + l_2) - y_2 \ ,$$

$$N > 0\,,\ if \left(N > 2\ , y_1 > \frac{y_2\,(N-2) - N\,(l_1 + l_2) + 2\,(l_1 + l_2 - 1)}{2 - N}\right)$$

Or,

$$y_1 > (l_1 + l_2) - y_2 \ ,$$

$$N > 0 \ , \ if \left(N > 2 \ , y_1 < \frac{y_2\,(N-2) - N\,(l_1 + l_2) + 2\,(l_1 + l_2 - 1)}{2 - N}\right)$$

Solving in terms of N:

$$\frac{1}{N\,(y_1 + y_2 - l_1 - l_2)} - \frac{N-2}{2N} > 0$$

$$if\ y_1 + y_2 - l_1 - l_2 < 0\ , \ N > 0 \ , \ if \left(y_1 + y_2 - l_1 - l_2 < 0\ , \ N < \frac{2(y_1 + y_2 - l_1 - l_2 + 1)}{y_1 + y_2 - l_1 - l_2}\right)$$

$$y_1 + y_2 - l_1 - l_2 < 0\ , \ N > 0 \ , \ if \left(y_1 + y_2 - l_1 - l_2 > 0\ , \ N > \frac{2(y_1 + y_2 - l_1 - l_2 + 1)}{y_1 + y_2 - l_1 - l_2}\right)$$

$$y_1 + y_2 - l_1 - l_2 > 0\ , \ N > 0 \ , \ if \left(y_1 + y_2 - l_1 - l_2 < 0\ , \ N > \frac{2(y_1 + y_2 - l_1 - l_2 + 1)}{y_1 + y_2 - l_1 - l_2}\right)$$

$$y_1 + y_2 - l_1 - l_2 > 0\ , \ N > 0 \ , \ if \left(y_1 + y_2 - l_1 - l_2 > 0\ , \ N < \frac{2(y_1 + y_2 - l_1 - l_2 + 1)}{y_1 + y_2 - l_1 - l_2}\right)$$

With these conditions, it is possible to show the decision-making terms for enterprises on entering or exiting the market or changing the strategy of enterprise location. In other words, assuming we have the number of streets, the location, and a certain location with the number of streets, it is possible to decide on production. Now, the next question is whether the results will change if the situation of enterprises is not symmetric, which will be examined in the following.

Demand if the Enterprises' Location is Asymmetric: A consumer located on point x on the empty street j is indifferent to purchasing from enterprises 1 and 2 when:

$$x_{12}^{j} = \frac{1}{2(y_1 - y_2 + l_2 - l_1)}\left[p_2 - p_1 + (y_2 - l_2)^2 - (y_1 - l_1)^2\right] + l_s \tag{25}$$

Without the loss of the whole subject, it is assumed that enterprise 1 is closer to the center ($l_1 - y_1 < l_2 - y_2$). Having the target function of convex transportation, the consumer who is closer to the center than the indifferent consumer purchases from enterprise 2, and the farther consumer will purchase from enterprise 1. If $x_{12}^{j} < 0$, all consumers on the empty streets purchase from enterprise 2 (Lijesen and Reggiani, 2013). The demand for all empty streets is obtained by multiplying the demand for one street by the total number of empty streets$\frac{2(N-2)}{N}$:

$$q_1 = \frac{2}{N}\left(1 - \frac{1}{2}(l_1 + l_2) + \frac{1}{2}(y_1 - y_2) + \frac{p_2 - p_1}{2(l_1 + l_2 - y_1 - y_2)}\right) - \frac{2}{2(y_1 - y_2 + l_2 - l_1)}(p_1 - p_2 + (y_1 - l_1)^2 - (y_2 - l_2)^2 + l_s)\frac{N-2}{N} \tag{26}$$

$$q_2 = \frac{2}{N}\left(1 - \frac{1}{2}(l_1 + l_2) + \frac{1}{2}(y_2 - y_1) + \frac{p_1 - p_2}{2(l_1 + l_2 - y_1 - y_2)}\right) + \left(1 + \frac{2}{2(y_1 - y_2 + l_2 - l_1)}(p_1 - p_2 + (y_1 - l_1)^2 - (y_2 - l_2)^2 - l_s)\frac{N-2}{N}\right) \tag{27}$$

Having equations 26 and 27, it is possible to present theorem 5:

Theorem 5: *If the situation of enterprises 1 and 2 (y_1 and y_2) establishes the following terms, then enterprise 1 increases its production by increasing the prices. Moreover, the increased number of streets leads to a decrease in the enterprise's production, and the production rate will be zero in N=∞.*

Proof: Differentiating equation 26, we have:

$$\frac{\partial q_1}{\partial p_1} = -\frac{2}{N(2l_1 + 2l_2 - 2y_1 - 2y_2)} - \frac{N-2}{N(y_1 - y_2 + l_2 - l_1)} > 0 \quad \Leftrightarrow \quad y_1 > -\frac{-y_2 + l_2 + 3l_1 - Nl_1 - Nl_2 + Ny_2}{-3 + N} \tag{28}$$

$$\frac{\partial q_1}{\partial p_1} = -\frac{2}{N(2l_1 + 2l_2 - 2y_1 - 2y_2)} - \frac{N-2}{N(y_1 - y_2 + l_2 - l_1)} > 0 \Leftrightarrow \quad y_2 > -\frac{-3y_1 + l_2 + 3l_1 - Nl_1 - Nl_2 + Ny_1}{-1 + N} \tag{29}$$

$$\frac{\partial q_1}{\partial p_2} = \frac{2}{N(2l_1 + 2l_2 - 2y_1 - 2y_2)} + \frac{N-2}{N(y_1 - y_2 + l_2 - l_1)} > 0 \Leftrightarrow \quad y_1 > -\frac{-y_2 + l_2 + 3l_1 - Nl_1 - Nl_2 + Ny_2}{-3 + N} \tag{30}$$

$$\frac{\partial q_1}{\partial p_2} = \frac{2}{N\,(2\,l_1 + 2\,l_2 - 2\,y_1 - 2\,y_2)} + \frac{N-2}{N(y_1 - y_2 + l_2 - l_1)} > 0 \iff y_2 > -\frac{-3y_1 + l_2 + 3l_1 - Nl_1 - Nl_2 + Ny_1}{-1 + N} \quad (31)$$

$$\frac{\partial q_1}{\partial N} = (2y_2^2 - 2l_1^2 + 4l_1y_1 + 2l_2^2 - 4l_2y_2 - 2y_1^2 - 6l_2y_1y_2 + 6l_1l_2y_2 - 6l_1l_2y_1 + 2l_1y_1y_2 + 3l_1^3 - 3l_2^3 - 3y_1^3 + y_2^3 - 9l_1^2y_1 - l_1^2y_2 + 3l_1^2l_2 - 3l_1l_2^2 + 9l_1y_1^2 + 3l_2^2y_1 + 7l_2^2y_2 + 3l_2y_1^2 - y_1^2y_2 + 3y_2^2y_1 - 5y_2^2l_2 - 3y_2^2l_1 - 3p_1y_1 - p_1y_2 + p_1l_2 + 3p_1l_1 + 3p_2y_1 + p_2y_2 - p_2l_2 - 3p_2l_1 + 2l_sl_1 + 2l_sl_2 - 2l_sy_1 - 2l_sy_2)/(l_1 + l_2 - y_1 - y_2)\,N^2\,(l_1 - l_2 + y_2 - y_1) \quad (32)$$

If N=∞, then $\frac{\partial q_1}{\partial N}$ =0.

These results, indicating that the increased number of streets leads to decreased production of enterprises, confirm Cournot's results in the N-seller game, where with an increase in the number of enterprises, the production of each enterprise is inclined toward zero, and the total production of the industry becomes closer to competitive production. The price selection in the symmetric status of enterprises will be investigated in the following.

3.4 Price Selection

We initiate price selection status from the enterprises' symmetric model on streets. Constituting profit function and then differentiating equation 23, it becomes possible to propose theorem 6:

Theorem 6: *If the location of enterprises is as $y_1 = y_2$ or $l_1 - y_1 = y_2 - l_2$, there would be a Nash equilibrium in prices, where $p_1 = p_2$.*

Proof: Constituting profit function and then the first rank conditions and solving and simplifying equation 23, we have:

$$p_1\left(\frac{1}{N(y_1 + y_2 - l_1 - l_2)} - \frac{N-2}{2N}\right) + \frac{2}{N}\left(1 - \frac{1}{2}(l_1 + l_2) + \frac{1}{2}(y_1 - y_2) + \frac{p_2 - p_1}{2(l_1 + l_2 - y_1 - y_2)}\right) + \frac{N-2}{2N} = 0 \quad (33)$$

Solving this equation and simplifying it would yield the following results:

$$p_1 = \frac{2p_2 + y_1(4l_1 + 4l_2 - 2) - 2y_2 + 2y_2^2 - 2y_1^2 + N\,(l_1 + l_2 - y_1 - y_2) - 2(l_1^2 + l_2^2 + 2l_1l_2) + 2l_1 + 2l_2}{4 - 2(l_1 + l_2 - y_1 - y_2) + N\,(l_1 + l_2 - y_1 - y_2)}$$

$$p_2 = \frac{2p_1 + y_2(4l_1 + 4l_2 - 2) - 2y_1 - 2y_2^2 + 2y_1^2 + N\,(l_1 + l_2 - y_1 - y_2) - 2(l_1^2 + l_2^2 + 2l_1l_2) + 2l_1 + 2l_2}{4 - 2(l_1 + l_2 - y_1 - y_2) + N\,(l_1 + l_2 - y_1 - y_2)}$$

$$\Rightarrow \quad p_1 - p_2 = -\frac{y_2(4l_1 + 4l_2 - 2) - y_1(4l_1 + 4l_2 - 2) - 4y_2^2 + 4\,y_1^2 - 2y_1 + 2y_2}{N\,(l_1 + l_2 - y_1 - y_2) - 2l_2 + 6 - 2l_1 + 2y_1 + 2y_2}$$

This difference will be zero with respect to $y_1 = y_2$ or $l_1 - y_1 = y_2 - l_2$. Economides (1989) studied the presence of equilibrium and optimality in a market for the distinction of products in terms of their diversity. It was shown that complete equilibrium of the sub-game was established in a three-stage game, where the enterprises entered in the first stage and diversity and price were selected in the second and third stages, respectively. In equilibrium, the products are symmetrically distributed based on their features, and equal prices will be offered. These results confirm the results obtained by Shahbazi and Salimian (2017), who showed that in a three-stage game where the firms entered in the first stage, selected diversity in the second stage and prices in the third stage, there was a subgame perfect equilibrium. In equilibrium, the products are distributed based on their symmetric specifications, and the same prices are presented. Moreover, Lijesen and Reggini (2013) achieved similar results and showed that if the situations of two firms were symmetric, they would both gain the same market share and receive similar prices.

Now, the issue of price selection in the asymmetric situation of enterprises will be studied. Theorem 7 is obtained by differentiating Equations 26 and 27.

Theorem 7: *If* $l_1 - y_1 < l_2 - y_2$, the *price Nash equilibrium will be presented and explained as follows:*

$$\begin{aligned} p_1 = -\Big(\Big(& y_1^3(N-3) + y_1^2(y_2(N-1) - N\,(3l_1 + l_2 + 1) + 11l_1 + 5l_2 - 6) \\ & - y_1(y_2^2(N-3) + 2y_2(N\,(l_1 - l_2) - l_1 + 3l_2) \\ & - N\,(3l_s + 3l_1^2 + 2l_1(l_2+1) - l_2^2) + 6l_s + 13l_1^2 + 2l_1(5l_2 - 6) - 3l_2^2) \\ & + y_2^3\,(1-N) + y_2^2(N(l_1 + 3l_2 + 1) - 5l_1 - 7l_2 + 6)\Big) \\ & + y_2\,\Big(N\,\big(3l_s + l_1^2 - 2\,l_1\,l_2 - l_2(\,3l_2 + 2)\big) - 6l_s - l_1^2 + 10l_1\,l_2 \\ & + l_2(11l_2 - 12)\,\Big) \\ & - (l_1 + l_2)\Big(N\,\big(3l_s + l_1^2 + l_1 - l_2(\,l_2 + 1)\big) - 6l_s - 5l_1^2 + 6l_1 \\ & + l_2(5l_2 - 6)\Big)\Big)/\big(3(y_1\,(N-3) + y_2(N-1) - N\,(l_1 + l_2) + 3l_1 + l_2)\big) \end{aligned} \tag{34}$$

$$\begin{aligned} p_2 = \Big(\Big(& y_1^3(N-3) + y_1^2(y_2(N-1) - N\,(3l_1 + l_2 - 2) + 5l_1 - l_2 + 6) - \\ & y_1(y_2^2(N-3) + 2y_2(N\,(l_1 - l_2) - l_1 + 3l_2) + N\,(3l_s - 3l_1^2 + 2l_1(2 - l_2) + \\ & l_2^2) - 6l_s + l_1^2 + 2l_1(6 - l_2) - 3l_2^2) + y_2^3\,(1-N) + y_2^2(N(l_1 + 3l_2 - 2) + l_1 - \\ & l_2 - 6)\Big) - y_2\,\Big(N\,\big(3l_s - l_1^2 + 2\,l_1\,l_2 + l_2(\,3l_2 - 4)\big) - 6l_s + l_1^2 + 2l_1\,l_2 + \\ & l_2(l_2 - 12)\,\Big) + (l_1 + l_2)\Big(N\,\big(3l_s - l_1^2 + 2l_1 + l_2(\,l_2 - 2)\big) - 6l_s - l_1^2 + 6l_1 + \\ & l_2(l_2 - 6)\Big)\Big)/\big(3(y_1\,(N-3) + y_2(N-1) - N\,(l_1 + l_2) + 3l_1 + l_2)\big) \end{aligned} \tag{35}$$

Proof: The profit functions of enterprises according to Equations 26 and 27 are as follow:

$$\pi_1 = (p_1\left(\frac{2}{N}\left(1-\frac{1}{2}(l_1+l_2)+\frac{1}{2}(y_1-y_2)+\frac{p_2-p_1}{2(l_1+l_2-y_1-y_2)}\right)-\frac{1}{(y_1-y_2+l_2-l_1)}\left((p_1-p_2+y_1^2+l_1^2-2y_1l_1-y_2^2-l_2^2+2y_2l_2)+l_s\right)\frac{(N-2)}{N}\right))$$

$$\pi_2 = p_2\left(\frac{2}{N}\left(1-\frac{1}{2}(l_1+l_2)+\frac{1}{2}(y_2-y_1)+\frac{p_1-p_2}{2(l_1+l_2-y_1-y_2)}\right)\right)+p_2\left(1+\frac{1}{(y_1-y_2+l_2-l_1)}\left((p_1-p_2+y_1^2+l_1^2-2y_1l_1-y_2^2-l_2^2+2y_2l_2)-l_s\right)\right)\frac{(N-2)}{N}\right)$$

Considering the above functions and the first rank condition, $p_1$and p_2 will be obtained after simplification, as explained in theorem 7.

Through analysis of a standard price-value game (first value and then price), Anderson et al. (1997) showed a unique strategy equilibrium in a two-stage game by consumers' density as log-concave. If the density is neither "too asymmetric" nor "too concave", the equilibrium locations will be closer, equilibrium prices will be lower, and density higher. In a symmetric density that is too concave, there is no symmetric equilibrium, although it might be in an asymmetric state.

The above results are in line with the results obtained by Lijesen and Reggiani with respect to $l_1 = l_2 = \frac{1}{2}$. In following, the location will be obtained solving the game through the backward induction method (after replacing equilibrium prices). First, the state where all enterprises have a symmetric situation toward the center will be studied.

3.5 Location Selection

Now we step back one stage and calculate the Nash equilibrium of the first stage of game concerning Nash equilibrium of the second stage. The decision variable in the first and second stages is location determination and price determination, respectively. Here, first the symmetric state of the enterprises' location will be initially examined, followed by their asymmetric state.

Theorem 8: *If* $N > \frac{4(l_1+l_2-1)}{2l_1+2\,l_2-1}$, *then there would not be a symmetric Nash equilibrium in the location.*

Proof: The profit function of enterprise 1 according to equation 23 will be as follows:

$$\pi_1 = p_1\left(\frac{2}{N}\left(1-\frac{1}{2}(l_1+l_2)+\frac{1}{2}(y_1-y_2)+\frac{p_2-p_1}{2(l_1+l_2-y_1-y_2)}\right)+\frac{(N-2)}{2N}\right) \quad (36)$$

Replacing the equilibrium prices and then the first rank conditions, the following results will be obtained:

$$y_1 = \frac{1}{8-4N}\Big(16y_2^2(N-2)^2 + 8y_2(2-N)\big(N(4l_1+4l_2-1) - 8(l_1+l_2-1)\big) + N^2(16l_1^2 + 8l_1(4l_2-1) + 16l_2^2 - 8l_2 - 47) - 16N\big(4l_1^2 + l_1(8l_2-9) + l_2(4l_2-9)\big) + 64\,(l_1^2 + 2l_1(l_2-2) + l_2^2 - 4l_2 + 3)\Big)^{\frac{1}{2}} - N$$

$$y_2 = \frac{1}{8-4N}\Big(16y_1^2(N-2)^2 + 8y_1(2-N)\big(N(4l_1+4l_2-1) - 8(l_1+l_2-1)\big) + N^2(16l_1^2 + 8l_1(4l_2-1) + 16l_2^2 - 8l_2 - 47) - 16N\big(4l_1^2 + l_1(8l_2-9) + l_2(4l_2-9)\big) + 64\,(l_1^2 + 2l_1(l_2-2) + l_2^2 - 4l_2 + 3)\Big)^{\frac{1}{2}} - N$$

After simplifying the above expressions, the enterprises' equilibrium location will be obtained as follows:

$$y_1 = \frac{N\,(2l_1^2 + l_1(4l_2-1) + 2l_2^2 - l_2 - 6) - 4\,(l_1^2 + 2l_1(l_2-2) + l_2^2 - 4l_2 + 3)}{2\,\big(N\,(2l_1 + 2l_2 - 1) - 4\,(l_1+l_2-1)\big)} \tag{37}$$

$$y_2 = \frac{N\,(2l_1^2 + l_1(4l_2-1) + 2l_2^2 - l_2 - 6) - 4\,(l_1^2 + 2l_1(l_2-2) + l_2^2 - 4l_2 + 3)}{2\,\big(N\,(2l_1 + 2l_2 - 1) - 4\,(l_1+l_2-1)\big)} \tag{38}$$

These results show that $y_1 - y_2 = 0$. If $N > \frac{4(l_1 + l_2 - 1)}{2l_1 + 2\,l_2 - 1}$, then $y_1 > l_1$, and the same condition will be established for enterprise 2. In the Lijesen and Reggiani model and for the special case of $l_1 = l_2 = \frac{1}{2}$, which is the maximum value, the result will be - $\frac{5}{2}$, which is out of range [0, l_i]. These results indicate that the symmetric status of enterprises is not Nash equilibrium when the number of streets is more than the above expression (and in the spoke model, more than two streets) since the enterprises should be located on the intended street. In this case, both enterprises will be located in the same place (center), and since each enterprise could increase its profit by deviation from its location, there is not symmetric Nash equilibrium. These results are consistent with the results obtained by Lijesen and Reggiani (2013).

4. Selection of Price and Location

In a game with sequential movement with two enterprises, the enterprises initially select their production level, and then the price will be selected at the next stage by observing the selected production level. If enterprises select price and location strategies, the following results will be obtained:

Theorem 9: *If there are just two streets of unequal lengths, and the location of enterprises is symmetric, $l_1 - y_1 = l_2 - y_2$, there is Bertrand Nash equilibrium in* *prices. However, symmetric Nash equilibrium doesn't exist in location, and one of the enterprises moves toward and the other escapes the center.*

Proof: Concerning the demand functions of enterprises 1 and 2 (according to equations 9 and 10) and the assumption that one of the enterprises is located on a street where the indifferent consumer is located, the following results will be achieved:

$-\frac{2}{3}l_1l_2 - \frac{2}{3}l_1y_2 + \frac{4}{3}l_2y_1 - \frac{2}{3}l_2y_2 - \frac{1}{3}l_1^2 - \frac{1}{3}l_2^2 - y_1^2 + y_2^2p_1 = \frac{4}{3}l_1y_1$

$+\frac{2}{3}l_1l_2 - \frac{4}{3}l_1y_2 + \frac{2}{3}l_2y_1 - \frac{4}{3}l_2y_2 + \frac{1}{3}l_1^2 + \frac{1}{3}l_2^2 - y_1^2 + y_2^2p_2 = \frac{2}{3}l_1y_1$

$q_1 = \frac{1}{6}l_1 + \frac{1}{6}l_2 - \frac{1}{2}y_1 + \frac{1}{2}y_2$

$q_2 = \frac{1}{6}l_1 + \frac{1}{6}l_2 + \frac{1}{2}y_1 - \frac{1}{2}y_2$

Thus,

$$\pi_1 = \frac{2}{9}l_1l_2y_2 - \frac{1}{18}l_1^3 - \frac{1}{18}l_2^3 - \frac{1}{6}l_1l_2^2 - \frac{1}{6}l_1^2l_2 + \frac{1}{9}l_1^2y_2 + \frac{1}{9}l_2^2y_2 \quad (39)$$

$$\pi_1 = \frac{2}{9}l_1l_2y_1 - \frac{1}{18}l_1^3 - \frac{1}{18}l_2^3 - \frac{1}{6}l_1l_2^2 - \frac{1}{6}l_1^2l_2 + \frac{1}{9}l_1^2y_1 + \frac{1}{9}l_2^2y_1 \quad (40)$$

$$\pi_2 = -\frac{2}{9}l_1l_2y_2 + \frac{1}{18}l_1^3 + \frac{1}{18}l_2^3 + \frac{1}{6}l_1l_2^2 + \frac{1}{6}l_1^2l_2 - \frac{1}{9}l_1^2y_2 - \frac{1}{9}l_2^2y_2 \quad (41)$$

$$\pi_2 = -\frac{2}{9}l_1l_2y_1 + \frac{1}{18}l_1^3 + \frac{1}{18}l_2^3 + \frac{1}{6}l_1l_2^2 + \frac{1}{6}l_1^2l_2 - \frac{1}{9}l_1^2y_1 - \frac{1}{9}l_2^2y_1 \quad (42)$$

Based on the obtained results, enterprise 1 is inclined toward becoming close to the center for a certain location from it or enterprise 2, and enterprise 2 is inclined to become far from the center with respect to a certain location from it or enterprise1. These results could be easily shown through the Hotelling linear model (1929). In the Hotelling linear model, if the length of streets is equal and the two enterprises are located at one point, then $p_1 = p_2 = 0$ is a unique equilibrium. When the two enterprises are located very close to each other, they begin breaking each other's prices, and equilibrium will not be achieved as a result of this price-breaking process. In this state, the enterprises are inclined toward distancing from each other, which is the same as Hotelling results that say:

In case some merchants make the transportation problematic instead of establishing road maintenance or enhancing the roads' reconstruction organizations, they have done a good job (Shy, 1995).

Now, the question arises whether the game will have equilibrium in the presence of streets of unequal lengths if the strategy of enterprises is the selection of both price and location.

Theorem 10: *In the Hotelling linear city game (with two unequal streets), when the strategy of enterprises is selecting both price and location, there would be no equilibrium for the game.*

Proof: To prove this theorem, we raise the question that how enterprise 1 would behave if the price and location of the competitive enterprise to enterprise 1 is given and enterprise 1 is allowed to perform locating again. Considering Equations 29 and 40, the results will be simply obtained:

$$\pi_1 = \frac{2}{9}l_1 l_2 y_2 - \frac{1}{18}l_1^3 - \frac{1}{18}l_2^3 - \frac{1}{6}l_1 l_2^2 - \frac{1}{6}l_1^2 l_2 + \frac{1}{9}l_1^2 y_2 + \frac{1}{9}l_2^2 y_2$$

$$\pi_1 = \frac{2}{9}l_1 l_2 y_1 - \frac{1}{18}l_1^3 - \frac{1}{18}l_2^3 - \frac{1}{6}l_1 l_2^2 - \frac{1}{6}l_1^2 l_2 + \frac{1}{9}l_1^2 y_1 + \frac{1}{9}l_2^2 y_1$$

The results show that the profit of enterprise 1 increases while moving toward enterprise 2, which is the principle of minimum differentiation. The obtained results confirm Bulow et al. (1985), who showed that when the products became homogenous, the profit level of all enterprises reduced to zero (the principle of minimum differentiation). On the other hand, if enterprises became so close to each other, there would be no equilibrium, and if enterprise 1 was exactly located at a point of enterprise 2, its profit would be zero, indicating that it would be better for enterprise 1 to distance from enterprise 2.

Kreps and Scheinkman (1983) presented a special framework (a special two-stage dynamic game), where the enterprises would select their production quantity at the first period (accumulated inventory). Since the production quantity was stabilized in the first period, and there was no possibility of its change, the enterprises would select the price in the second period. They showed that the selected values by the enterprises in the first stage and the selected prices in the second stage were exactly equal to Cournot's results. They also showed that in some markets, in those games where two enterprises selected their production quantity in the first stage and then determined their prices in the second stage, SPE was a game according to the production quantity and price in a single-stage game of Cournot structure, where the enterprises just determined their production quantity.

Economides (1986) showed that firstly, the principle of minimum differentiation was not established in Hotelling price-location, since the complete sub-game equilibrium in long term did not exist with the minimum differentiation of products. Secondly, it opposes the principle of maximum differentiation and is totally incorrect. Although maximum differentiation of product is held for most convex transportation cost functions, there is a wide range of utility functions whose complete equilibrium location is represented by inner points of product space. Now, we examine the effect of the length and number of streets on the profit and situation of enterprises.

The Length and Number of Streets

Since the length and number of streets are the main issues in determining the profit and location of enterprises, and specifically effective on enterprises' profit, it is possible to achieve the following results from the profit function:

Theorem 11: *With the increased length of streets, the enterprises' profit increases if:*

$$y_1 + y_2 - \sqrt{p_1 - p_2} - l_2 \le l_1 \le y_1 + y_2 + \sqrt{p_1 - p_2} - l_2 \tag{43}$$

And

$$y_1 + y_2 - \sqrt{p_1 - p_2} - l_1 \le l_2 \le y_1 + y_2 + \sqrt{p_1 - p_2} - l_1 \tag{44}$$

Proof: According to equation 36:

$$\frac{\partial \pi_1}{\partial l_1} = \frac{p_1\,(y_1^2 + 2y_1(y_2 - l_1 - l_2) + y_2^2 - 2y_2(l_1 + l_2) - p_1 + p_2 + l_1^2 + 2l_1l_2 + l_2^2\,)}{N\,(1 - N)\,(y_1 + y_2 - l_1 - l_2)^2}$$

The denominator is always negative for N ≥ 2. For the fraction to be positive, it is sufficient to have a negative nominator, leading to the following result:

$$y_1 + y_2 + \sqrt{p_1 - p_2} - l_2 \le l_1 \le y_1 + y_2 - \sqrt{p_1 - p_2} - l_2$$

Or

$$y_1 + y_2 - \sqrt{p_1 - p_2} - l_2 \le l_1 \le y_1 + y_2 + \sqrt{p_1 - p_2} - l_2$$

The following results will be obtained from differentiation from the profit function of enterprise 1 with respect to l_2:

$$y_1 + y_2 + \sqrt{p_1 - p_2} - l_1 \le l_2 \le y_1 + y_2 - \sqrt{p_1 - p_2} - l_1$$

Or

$$y_1 + y_2 - \sqrt{p_1 - p_2} - l_1 \le l_2 \le y_1 + y_2 + \sqrt{p_1 - p_2} - l_1$$

It is obvious that just the second condition could be established. Thus, if these conditions hold, enterprise 1 gains more profit concerning the increased length of streets. Assuming the single-product enterprises, Manez and Waterson (2001) investigated the outcomes of horizontal and vertical products in market structure and showed that: 1. The insider price elasticity is decreasing relative to quality, 2. The cross-price elasticity is decreasing relative to quality, and 3. Mark-ups are increasing relative to quality. These results could be expressed as the maximum differentiation where the distance from ideal feature is to be far from it. The last theorem is presented as follows:

Theorem 12: *With an increase in the number of streets, the enterprises select the farther location (maximum differentiation) unless under certain conditions, where the enterprises select the location closer to the center (minimum differentiation).*

$1 - l_1 \leq l_2 \leq \frac{3}{2} - l_1$

Proof: According to Equations 37 and 38, we have:

$$\frac{\partial y_1}{\partial N} = -\frac{6\left(2l_1^2 + l_1(4l_2 - 5) + 2l_2^2 - 5l_2 + 3\right)}{\left(N(2l_1 + 2l_2 - 1) - 4(l_1 + l_2 - 1)\right)^2} = \frac{\partial y_2}{\partial N} = 0$$
$$\Rightarrow N = \infty \text{ or } 2l_1^2 + l_1(4l_2 - 5) + 2l_2^2 - 5l_2 = -3 \qquad (45)$$
$$\frac{\partial y_1}{\partial N} = -\frac{6\left(2l_1^2 + l_1(4l_2 - 5) + 2l_2^2 - 5l_2 + 3\right)}{\left(N(2l_1 + 2l_2 - 1) - 4(l_1 + l_2 - 1)\right)^2} = \frac{\partial y_2}{\partial N} \leq 0$$

Since this equation will be positive just under the above conditions, with an increase in the number of streets, the enterprises select farther points from the center as their optimum location. In case the above conditions hold, with an increase in the number of streets, the enterprises select the closer points to the center as their optimum location. The results of this section are similar to vertical differentiation and the results of Manez and Waterson, as discussed in theorem 12. Vandenbousch and Weinberg (1995) extended the vertical differentiation model of Shaked and Sutton (1982) and Moutry (1988), expanding it to two dimensions and analyzing price and quantity competition. They showed that despite the vertical differentiation model, the enterprises were not inclined toward maximum differentiation. Although this solution might hold under certain conditions, when a wide range of location in every dimension is equal, the max-min product differentiation will occur. In equilibrium, the enterprises are inclined to select the location, which shows maximum differentiation in one dimension and minimum differentiation in another.

5. Conclusion, and Recommendation

One of the key and effective issues in the implementation and optimization of projects is to determine an appropriate location for them, which requires recognition and identification of the basic parameters and structures in the studied areas. The establishment of any urban element in a specific spatial-physical location of a city is subject to certain principles, rules, and mechanisms that if respected, the success and functional efficiency of that element will be greater at the exact location; otherwise, there will be many problems. Locating is the selection of a location for one or more centers while considering other centers and existing constraints to optimize a special purpose. This objective could be cost of transportation, gaining more profit, providing fair services to customers, taking the largest market share, etc. These issues and the review of literature on locating show that most studies have used simplification assumptions for locating. One of the main assumptions is about circular cities and linear cities with streets of equal length. It is obvious that in the real world, the length of streets is not necessarily equal and may be affected by many factors. This paper expanded Chen and Riordan's initial model and its extended version by Lijesen and Reggiani considering streets of unequal lengths. One of the main results of this paper is keeping the results of Lijesen and Reggiani with respect to the streets' length of $\frac{1}{2}$ in all cases, indicating that Lijsen and Rejiani's model is a special case of a more general model of streets of unequal length. In addition to confirming all results of Lijesen and Reggiani for streets of unequal length, the following results have been obtained:

1. Concerning theorem 1, the spoke model is a certain form of the streets with unequal length model (just in special cases where all streets are of equal length and almost $\frac{1}{2}$).
2. Concerning theorems 3, 4, 5, and 12, the "increased number of streets is effective on location and profit of enterprises".
3. Concerning theorem 5, "with an increase in the number of streets, the production of enterprise would decrease, and when the number of enterprises inclines toward infinity, it becomes closer to completion consequence).
4. Concerning theorem 1, "in the streets with unequal length model, there is a unit Nash equilibrium in prices, and this equilibrium price has a direct relation with the number of streets".
5. Concerning theorem 6, "if the distance of enterprises from the center is equal, there is Nash equilibrium in prices where the enterprises demand for the same price".
6. Concerning theorem 3, "the increase in the number of empty streets of enterprise, the enterprises select locations farther than the center".
7. Concerning theorem 9, "if there are just two streets of unequal lengths and the location of enterprises is symmetric, Bertrand Nash equilibrium would exist in prices, but symmetric Nash equilibrium doesn't exist".
8. Concerning theorem 10, "in case there are two streets of unequal lengths and the strategy of enterprises is the selection of both price and location, there would be no equilibrium for the game".

Briefly speaking, since cities consist of streets of unequal lengths in the real world, the enterprises could increase their prices with an increase in the number and length of streets, becoming farther from center, and increasing the price by the other enterprise. Moreover, the enterprises are recommended to move toward the center with an increase in the number of streets.

In the end, since there are many simplifying assumptions in locating models, and the assumption of uniform distribution of consumers is held, it is recommended to replace this assumption with better realities, such as statistical distributions. In addition, various issues, such as current location of enterprises, including future changes and revolutions, such as threats, opportunities, demand growth, lack of sufficient information by the enterprises, environmental factors such as political pressures, disturbance of balance, and many other factors could be effective in enterprises' locating. Thus, it is recommended to use these factors within the framework of imaginary variables in future studies.

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